

The University of Chicago

FUNCTIONAL THINKING
AND THE STUDY OF FUNCTIONAL
RELATIONS IN ELEMENTARY
ALGEBRA

A PART OF A DISSERTATION SUBMITTED TO
THE FACULTY OF THE DIVISION OF THE
SOCIAL SCIENCES IN CANDIDACY FOR THE
DEGREE OF DOCTOR OF PHILOSOPHY

DEPARTMENT OF EDUCATION

1935

By

ALFRED JOSEPH PYKE

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FOREWORD

The following essential portion of the thesis consists of excerpts from the complete dissertation on file only in the University of Chicago Library. These excerpts are taken from:

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CHAPTER I

THE STATEMENT OF THE PROBLEM

Preliminary Statement

Leaders in mathematical science and education have recommended that the curriculum in mathematical education in the elementary and secondary schools should be unified about the concept of the function and that the curriculum should provide opportunity wherein the pupils may engage continually in functional thinking as they gain their knowledge of mathematics. This recommendation is concerned both with the material of the curriculum and with its organization and presentation. If the thought of functional dependence is to be the central theme throughout the presentation of material in arithmetic, algebra, geometry, and trigonometry, then it should result in the development of certain well defined mental attitudes and modes of thought within the individual. The unifying factor in the presentation of the several branches of mathematics is the development of a general appreciation of relationships which is to dominate all acquisition of knowledge by the student.¹

¹F. Klein, Vortrage uber den Mathematischen Unterricht an den hoheren Schulen. Leipsic: Teubner, 1907.

W. Lietzman, Methodik des Mathematischen Unterricht, Teile I, II, III. Leipsic: Quella and Meyer, 1916, 1919, 1924, 1926.

C. Méray, Nouveau précis d'analyse infinitésimale. Paris: F. Savy, 1872.

J. Tannery, Notions de Mathématiques. Paris: C. Delagrave, 1903.

J. Perry, Address: "The Teaching of Mathematics," delivered at Glasgow before the British Association, 1901. Educational Review (1902), p. 158.

T. P. Nunn, The Teaching of Algebra (including Trigonometry). London: Longmans, Green & Co., 1914, 1919, 1927.

E. H. Moore, "On the Foundations of Mathematics," Bulletin of the American Mathematical Society, IX (1903), 402-24.

National Committee on Mathematical Requirements, "The Function Concept in Secondary Education," The Reorganization of Mathematics in Secondary Education, pp. 64-73. New York: Mathematical Association of America, 1923.

Organized society is much concerned with the variable, with the function, with change, and with rates of change. To be convinced of this, one need but notice the increasing prominence given to tables of statistical data and to the graph in the daily paper as it conveys to us its story of the changes of an active and intimately related world. The following evidence of the extensive use of tabular representation of functional relations was taken from one copy of the Chicago Daily Tribune, January 13, 1931. Four other copies of different dates were examined and the results were similar. There were 2,100 stock quotations, 5 financial reports, 134 theater announcements, 5 daily radio programs, 45 athletic reports, 49 market reports, 20 railway or motor bus time tables, 2 weather reports, and so many other minor reports that it was difficult to complete the total of such representations of functional relation. An examination of one year's issue of the Chicago Daily Tribune, 1929, revealed interesting information concerning the use of the graph in the distribution of daily news. During the year, 372 graphs were published; of these, 363 dealt with business reports from the Chicago Tribune Survey, one of which appeared each day, and 9 occurred in the advertisements. Of the 372 graphs, there were 1 pictograph, 224 bar graphs, 9 circular graphs, and 139 line graphs. This is evidence that the publishers of a paper whose circulation approximates a million copies each day expect its readers to appreciate, understand, and interpret relations between varying magnitudes. Few formulas other than those dealing with interest and rates of interest were published in this year's issue of the Chicago Daily Tribune.

We conclude that pupils should graduate from the schools fitted to understand and appreciate such expressions of relations between related varying magnitudes and functional dependence. If the affairs of our modern world are guided by men who are continually engaged in interpreting and expressing relations between varying quantities, and if the pupil is to take his part in this world as a contributing member, then he should be equipped with an enquiring and critical attitude and with appropriate modes of thought regarding change, changing quantities, and the laws relating them. Those students who graduate to the technical schools and colleges, to the professional schools and colleges, and to the universities are unable to proceed with their advanced studies in many of the sciences and kindred subjects until they possess the basic ideas of the calculus. Practically all of the higher institutions of learning recognize this, and so present introductory courses in the calculus as early as possible in their programs.

Statement of the Problem

It is proposed in this thesis:

- I. To define the concept of function as used in modern mathematics.

To do this, the development of the idea of function is traced briefly and various definitions are offered expressing the thought as the modern concept developed. This is necessary before one can select that phase of the concept and that mode of expression best suited for presentation to young pupils. Chapter II includes this definition expressed by means of:

 - A. A brief survey of the development of the present idea of function.
 - B. Definitions due to Bernoulli, Dirichlet, Moore.
 - C. Many illustrations.
 - D. A concise resume of current modes of expression of this concept.
- II. To analyse that method of thinking known as Functional Thinking by defining the variable and the function as used generally and as used by the specialist in mathematics.
- III. To design a series of lessons, covering two years' work in Elementary Algebra for Grades IX and X of the secondary school, in which the concept of function shall be the central and unifying principle.

This series of lessons is described in Chapter IV and in the Appendix to this report. The pupils to whom these lessons were presented were being taught the course of studies for Grades IX and X as prescribed by the Department of Education for the Province of Saskatchewan, Canada; hence the topics for the year's work were already chosen. Since this study is primarily concerned with a method of presentation by which the algebra may be co-ordinated with the broader principles of the function, these authorized topics were accepted as best for this experimental course in that it was considered unwise to disturb the students' course of studies more than was necessary. The Dirichlet concept of the function as described in Chapter II of this report is generally emphasized throughout this presentation,¹ but the extended definition by Moore is kept constantly under review.

The characteristic features of the series of lessons are:

- A. Each lesson contributes to the general theme of function.
- B. The pupils study arithmetical situations and under

¹See p. 14.

guidance participate in an algebraic situation before they introduce the algebra.

- C. Throughout the whole series of lessons, the pupils are urged to investigate number relations, by means of which they are guided to appreciate the meaning and use of elementary algebra.
 - D. Correct attitudes and appropriate modes of thought for the subject are emphasized continually as the pupils learn their algebra, organized about the variable and based upon functional thinking.
- IV. To teach this series of lessons to a group of pupils of average ability commencing their studies at a good secondary school.

The students selected were in attendance at the Nutana Collegiate Institute, Saskatoon, Saskatchewan, Canada. Mr. A. W. Cameron, B. A., is the Principal of the Nutana Collegiate Institute, and Mr. M. J. Sexsmith, B. A., is senior mathematical master. A more detailed description of this class and the control classes will be found at the beginning of Chapter V.

- V. To administer certain standard tests in algebra and in functional thinking to the pupils of the experimental class and to pupils of certain control classes with a view to estimating and comparing progress in elementary algebra and functional thinking. The testing program is described in Chapters V and VI.

The tests used were:

- A. Breslich Algebra Survey Test, First Semester, Form A, applied in December, 1932.
 - B. Breslich Algebra Survey Test, Second Semester, Form A, applied in June, 1933.
 - C. Breslich Functional Thinking Test (as described), applied in June, 1933.
- VI. To summarize the evidence of success in the results of the teaching and testing program. This resume is presented in Chapters VII and VIII.

CHAPTER II

THE CONCEPT OF FUNCTION

In this chapter it is proposed to explain the meaning of a function of one variable by presenting a very brief summary of the origin of this idea; to review several definitions of this term as the modern concept evolved; to define a function of one variable as accepted in modern mathematics; to illustrate, from our immediate environment, a few of the many forms by which this concept is expressed; and, finally, to point out the relation of this thought to introductory algebra.

The idea of function occupied an important place in the methods of investigation long before it received its name in the field of mathematics. The history of any one of man's major activities, especially those pertaining to science, portrays many instances of functional thinking and leads one to conclude that such mental processes are responsible for most forms of progress. Since this study is concerned with mathematical education, most of the observations which follow are taken from this phase of thought, wherein the term function received its name and form.

The concept of function preceded the invention of the simple formula and the ingenious devices employed in early civilization for measuring, surveying, navigating, trading, computing, and so forth. For example, Ahmes (1650-1550 B.C.) was engaged in functional thinking as he pronounced that the area of a triangle of base 'b' and altitude 'h' was $\frac{1}{2}bh$ and that the area of a circle with diameter 'd' was $(d - d/9)^2$; he used a modern form of expression of a functional relation when he prepared one of the earliest known forms of a correspondence between certain angles and the trigonometric ratios which is now regarded as the origin of modern trigonometric functions. Euclid's Elements have their origin in functional thinking and the material for the elements consists of functional relations. The Greeks delighted in philosophy; their fondness for geometry and their consequent tendency to represent number and number relations by segments of lines, areas, and space led them to express their observed functional relations in terms of geometry. For example, "similar triangles are to one another as the squares on their homologous sides" implies a tremendously important functional relation and is expressed in a concise and appropriate form characteristic of this period in mathematical

thought. For centuries, functional relations were expressed explicitly and implicitly by rule, by arithmetic, by a growing algebra and trigonometry, and by many other methods; but the early association of functional relations with geometry, due to the powerful influence of the Greeks, had a very long and far reaching influence upon it. It was not until the beginning of the seventeenth century, when Rene Descartes (1596-1650)¹ gave us a method for combining algebra with geometry, that functional relations were recognized as such quite apart from their appearance in algebra, geometry, arithmetic, rules, or statements.

As knowledge increased, many exact number relations were discovered and expressed in various forms characteristic of mathematical thought; truths from the combined fields of arithmetic and algebra, from geometry, and from trigonometry blended with one another to form material for modern mathematical analysis. Toward the end of the seventeenth century, when generalizations and formulas were being examined for common properties, the concept of function, which had been a hidden but forceful factor in creative mathematics throughout the ages, presented itself in a very humble manner and claimed attention and definition.

The name, function, was first used explicitly by G. W. Leibnitz in 1694 in a geometric sense, to describe the length of a line segment which varied in accordance with a certain given law.

A few years later, in 1718, John Bernoulli used the term, function, with almost its modern significance. He described the function as a symbol composed in any manner of the variable and of constants. Hence, from the beginning, the concept of function is associated with a double meaning, the ordinate of a curve in geometry, due to Leibnitz, and the algebraic formula, due to Bernoulli. It appears to have revealed itself in the highways of mathematical thought as a quiet tribute to that splendid group of mathematicians who, inspired by Descartes, brought about the union of the sciences of algebra and geometry during the seventeenth century.

In 1730, John Bernoulli classified functions as algebraic and transcendental.

The present general notation, $f(x)$, representing a function of a variable, x , was introduced by Leonard Euler in 1734.

As stated, the essential characteristic of the early definition of function as given by John Bernoulli is the existence of some kind of formula which is used to compute the value of the dependent variable when that of the independent variable is known.

¹Rene Descartes, Geometrie, Livre premier. Paris, 1637.
Trans. Smith and Latham. Chicago: Open Court Publishing Co., 1925.

This thought still predominates in the schools and colleges as the students are introduced to the study of functions. Four distinct ideas seem to be necessary to understand this definition:

1. The meaning of variable, which constitutes an essential abstraction in elementary algebra.
2. An appreciation of related variables and their representation.
3. An appreciation of variables related by an analytic formula.
4. The general description of any function by symbol $f(x)$.

Until recently, these ideas were presented separately in different divisions of mathematical studies, but there is a tendency toward a unified presentation of the concept of function in elementary courses in mathematical analysis leading to the study of the calculus and to the more advanced theory of functions.

The idea of function was early identified with an analytic formula and with the ordinate of a curve and remained so for a long time. As a result, two problems became closely associated with the development of this concept, viz.:

1. Having given any analytic formula $f(x)$, the equation $y = f(x)$ determined a curve which constituted the locus of the variable point $P(x,y)$ where x and y satisfied the equation $y = f(x)$. Such a curve could be plotted in the usual way by choosing arbitrary values for the abscissa x and computing y from the formula for any desired number of values.

2. Having given an arbitrary geometric curve, it was required to devise an analytic formula which should describe the functional relation (if any) existing between the ordinate and abscissa of any point $P(x,y)$ on this curve.

This second problem proved to be very difficult. Its solution occupied the attention of leading mathematicians for many years. The names of Fourier (1768-1830) and Dirichlet are associated with a series of valuable researches seeking a solution to this second problem. As a result of their investigations, both men contributed to the generalization of the original definition of function. With Fourier, the function came to represent a correspondence between two related variables such as could be represented by the simultaneous variation of the ordinate and abscissa of a point on an arbitrary curve. This thought of correspondence prepared for the more general concept of functional dependence.

The following definition of a function of one variable serves to show us just what the mathematicians of this period meant by the function. This definition is usually attributed to Dirichlet, although it is slightly more general than that published

by Dirichlet in Dirichlet Werke I, page 132.¹

1. Let a and b be two fixed real numbers.
2. Let x be a variable which assumes in turn all values between a and b .
3. Let y be a second variable whose value depends upon the value assigned to x in such a manner that to every value of x between a and b there corresponds a definite value of y . Then y is said to be a function of the variable x in the interval from a to b .

The important feature of this second definition of function of one variable is that to every value of x in the interval (a,b) there shall correspond a unique and determined value of y , and that nothing is specified concerning the nature of the functional relation which obtains between the variables x and y . It is, in this particular lack of restriction upon the nature of the functional relation, that Dirichlet extended the concept of function. We infer that a function $f(x)$ may be defined for each value of x which enters into our discussion, even though we are not able to satisfy the claims of the earlier analysts that functional dependence be expressed by means of an analytic formula or be representable by the ordinate of some geometric curve. If it should happen that the dependence of y upon x can be recorded in this manner, it still satisfies the more general conditions imposed in the second definition, and thus all functions defined by Bernoulli and later extended by Fourier are included in Dirichlet's definition. In addition, many other types of functional relations which occurred with the development of mathematical analysis are included under this more general definition.

Let us sum up our observations concerning the definition of a function of one variable at this stage of its development.

1. All functions of one variable involve a set of corresponding numbers, that is, a function is a sort of numerical table wherein to each value of the independent variable in a certain specified domain there corresponds a definite value for the dependent variable.
2. Some functional relationships may be expressed by analytical expressions which are used as formulas to compute values of the dependent variable when values of the independent variable are known.
3. Some functional relationships may be represented by the

¹G. L. Dirichlet, "Ueber die Darstellung ganz willkürlicher Functionen durch Sinus und Cosinus reihen," Werke, I, 132. Berlin: Georg Reimer, 1889.

ordinate and abscissa of certain points on a plane surface. The locus of such points is called the graph of the function. It usually exhibits many of the characteristic properties of the function.

4. The following illustrations indicate the scope and limitation of the simple rule, the formula, the table of corresponding values, and the graph when used to express functional relations.

- a) Two boys are seated in different rooms. At the sound of a bell they each write a number chosen at will. At the close of such an exercise, we would have two sets of numbers forming a simple correspondence. This is the one essential feature of a functional relation.
- b) An automobile is 150 miles from Chicago and is traveling uniformly toward the city at 30 miles an hour. Its distance from the city is given by the equation $d = 150 - 30t$ where 't' represents the time in hours and 'd' the distance from the city in miles. Here we have a simple rule expressible by an equation, and we may form from this a table of corresponding values for t and d from which to plot a graph in the usual manner.
- c) A self-registering thermometer records the temperature of the day and presents a continuous graph of temperature and time. This situation has a mechanical device which portrays the corresponding values of temperature and time as a continuous graph. There is no simple rule or formula relating time and temperature.
- d) When x takes the value of the rational numbers in the range 1-4 let the value of y be 1 and when x takes the value of the irrational numbers in the same range let the value of y be 2. For each value of x in its range 1-4 there is a corresponding value for y but such a relation could not be described readily by formula nor pictured by a graph.

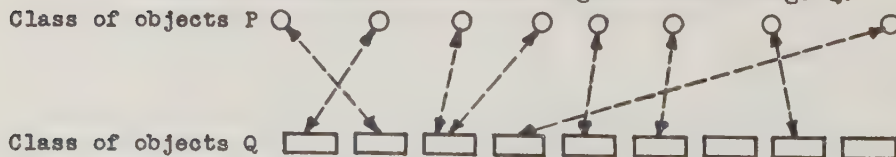
There is still one further generalization of this concept of function. This is the outcome of quite recent investigations in Mathematical Analysis due to Professor E. H. Moore, the Department of Mathematics, the University of Chicago.

Probably one should consider the present attitude toward the function as quite an open one and accept this last definition as an expression of a current point of view which indicates a present status and a trend of thought toward this exceedingly

important and central theme. The fact that the need of such a broad unifying concept was not appreciated until the seventeenth century and that it occupies so prominent a place in mathematics today may be accepted as evidence that attitudes and modes of thought change and so force a new alignment of earlier established truth.

In defining and illustrating this modern point of view of a function of one variable, I have consulted notes taken in a course of lectures on "Theory of Function of a Real Variable" given at the University of Chicago in the summer of 1927 by Professor R. W. Barnard. This definition is much the same as that attributed to Dirichlet, with this difference, that no reference is made to the fact that x and y shall be real numbers. The last generalization assigns to x and y as their domain any set of objects, but if the set of objects happens to be real numbers this last definition of function reduces to that given by Dirichlet. Again the concept is extended to include relations which were not recognized as functional relations in the earlier definitions. This last extension of the concept of a function of one variable is defined as follows:

1. A variable is a symbol p which in the course of a certain discussion may be used to designate any one of a given set of objects. The set of objects is called the range of the variable p . The totality of the set of objects in the range is frequently denoted by P and is described as a class of objects.
2. Function of one variable.
 - a) Let p be a variable whose range is a class of objects P .
 - b) Let q be a second variable whose range is a second class of objects Q .
 - c) Let the variable q be so related to the variable p that when p is given a definite value in the range P , q has a definite value on the range Q . Then q is said to be a function of p and the functional relation is expressed by the notation $q = f(p)$.
3. The following diagram illustrates several characteristics of a "Function on the range P to the range Q ."



4. There are three essential components to this concept:
 - a) A class of objects P each of which may be designated

by the independent variable p .

- b) A class of objects Q each of which may be designated by the dependent variable q .
- c) A form of correspondence, not necessarily one to one, which characterizes a relation between p and q such that when p is given a specified value or place on the range P , the functional value q corresponding to p is expressed by the notation $q = f(p)$ on range Q .

This correspondence may be described in a variety of ways.

It may be expressed in the words of a physical law relating two variables, by a tabular array of statistical data arising from a scientific investigation, by the mechanical graphical record of the variation of two varying magnitudes, by real numbers and points on a line, by an analytic formula, and by an equation; in fact, any scheme for setting up or recording a correspondence between two classes of objects constitutes a function in that the essential feature of a function is a correspondence.

The following illustrations are offered that they may assist the reader to appreciate functional relations and thereby add to a better interpretation of the foregoing definition.

I. ILLUSTRATIONS OF FUNCTIONAL RELATIONS FROM MATHEMATICAL SUBJECTS

A. Illustrations from Arithmetic:

1. On the positive integers to their squares. This refers to the functional relation given in the table of squares of the positive integers.
2. On time to the amount of money accumulated in investment. This refers to the correspondence between time and the amount accumulated by a given sum of money earning interest at a given rate. This correspondence is found in a set of interest tables.

B. Illustrations from Elementary Algebra:

On one set of real numbers to a second set of real numbers. This describes the correspondence underlying an algebraic expression, such as, $x^2 + 5x + 7$.

C. Illustration from Plane Trigonometry:

On angles to real numbers. This describes the functional relation between the angles and any one of the trigonometric ratios, such as the table of sines.

D. Illustration from Plane Geometry:

On pairs of distinct points to lines. This describes the correspondence implied in the statement "Two distinct points determine a line."

E. Illustration from Analytical Geometry:

On pairs of real numbers to points in a plane. This

refers to the correspondence about which analytical geometry is constructed.

F. Illustration from Elementary Calculus:

On real numbers to real numbers. This describes the correspondence for the calculus. When we state that $\sin x$ is a function of x we have such a correspondence in mind.

G. Illustration from Measurement:

On real numbers to the measure of a quantity in terms of the unit. This refers to a functional relation resulting from measurement. The measure of a quantity is described in terms of the number of units of measurement.

Let us sum up the description of methods of representing functional relation:

1. By words, which includes precise scientific laws; rules not recognized as scientific laws; and implied functional relations.

2. By table, including numerical correspondences and those not numerical.

3. By graphs of various types, such as the pictograph, the bar graph, the circular graph, and the line graph.

4. By formula, including the special form of the equation.

In this thesis it is proposed to describe a plan whereby one may present elementary ideas concerning varying magnitudes, the variable, and the function as the pupils review number relations in arithmetic and commence their studies in algebra. Most of the illustrations in the lessons make use of that special class of functional relations where P is a class of integers and Q is a class of real numbers. The following outline describes an approach to elementary algebra while placing a continued emphasis upon the study of functional relations.

1. The pupils are taught early the meaning of a correspondence.

2. The pupils learn that a function is a correspondence. The term "function" is not employed in the early presentation.

3. A correspondence may be expressed by verbal statement, table, graph, and formula or rule of combining numbers.

4. The early formulas presented for class discussion and consideration originate in generalized arithmetic.

5. The study of these formulas constitutes the introduction to elementary algebra.

CHAPTER III

FUNCTIONAL THINKING

The Independent Variable

An appreciation of the variable rests upon an appreciation of change in the changing quantities of our physical world. It is frequently necessary to attend to changing quantities and this requires an observation of a series of different states of existence for the quantities which are identifiable as the history of the changing quantity. In order to make or think comments concerning the class of quantities which shall be applicable to each and every state in the aggregate, the variable has been invented. The following definition of the variable is due to E. H. Moore and given by R. W. Barnard in his lectures on Theory of Functions of a Real Variable in the Summer Course at the University of Chicago in 1928.

"A variable is a symbol p which, in the course of a certain discussion, may be used to designate any one of a given set of objects. The set of objects is called the range of the variable. The totality of the set of objects in the range may be denoted by P and is usually referred to as a class of objects."

Thus the essentials of a variable are:

1. A changing quantity, which is not necessarily measurable.
2. An assemblage of particular values for the quantity which is to be called a class of quantities.
3. A symbol which is to designate any one of the elements of the class of quantities.

For example, when we refer to the temperature of a room, the time of day, the distance from home, the fragrance of flowers, the mayor of the city, the honesty of men, and so forth, we may have in mind a single, particular, specific value of the changing quantity described by the terms, temperature, time, distance, fragrance, mayor, honesty; or we may be thinking of a set of values associated with the variables, temperature, time, distance, fragrance, mayor, and honesty. The two points of view are entirely different. The mental attitude toward a variable is that of a one-dimensioned survey of a succession of objects; it demands a reviewing attitude of mind and a special characteristic method

of thinking toward changing quantities. Such an attitude of mind is a first essential in functional thinking.

The Mathematical Correlate to the
General Concept of Variable

When the varying magnitudes are measurable, the class of objects under observation are replaced by their measures in terms of units appropriate to the objects. The fundamentals of the student's attention are changed from quantities to measures of quantities and henceforth the student attends to a class of numbers and is no longer interested in their origin as measures of quantities existing in the physical world. The term "variable" originated in discussions with classes of numbers and has since been extended to include the more general class of objects.

The definition of variable is attributed to Weierstrass, who first used the term in a course of lectures in mathematics given in Berlin between 1861 and 1865.

"A real variable is a symbol which represents the different elements of an assemblage of real numbers. Each of these elements is one of the values which the variable can take.

The assemblage itself constitutes the domain of the variable."¹

The following definition of a variable is used by Dirichlet when he presents a generalization for the dependent variable or function in Dirichlet Werke in 1889.

"A variable is a symbol x which, in the course of a certain discussion, may be used to designate any one of a given set of numbers. The set of numbers is called the range of the variable. The totality of the set of numbers in the range may be denoted by X and is referred to as a class of numbers."²

In current textbooks in mathematics, one finds the variable defined as a quantity, an aggregate of values for a given quantity, and as a symbol. It would appear that the different writers choose to emphasize certain aspects of the idea of variable rather than the complete concept. In this article the term "variable" will be used as a symbol with the connotation explained above.

T. P. Nunn describes the variable in The Teaching of Algebra (Including Trigonometry) as follows:

"In the statement 'The King of England is a constitutional monarch' the element 'The King of England' is a variable in

¹J. Molk, Encyclopedie des sciences Mathematiques pures et appliquees, Tome 2, Vol. I, Fascicule 1, p. 16. Paris, 1909.

²Op. cit., I, 132.

exactly the same sense as V^- in the formula $V^- = Ah$."1

"What is not usually known is that variables are almost as common outside mathematics as within."2

"The prudent teacher will therefore, in the interests of clear understanding and economy of effort, present the technical use of variables in mathematics not as a new thing but as merely a modification of linguistic uses which the pupil mastered in principle at the mother's knee."3

"In denoting any member of the first, second or third class by the symbols A, B or C, Vieta followed [says Wallis] the custom of lawyers who 'put cases in the name of John an-Oaks and John a-Stiles or John a-Downs and the like (by which names they mean any person indefinitely who may be so concerned).'"4

"The invention of variables was perhaps the most important event in human evolution. The command of their use remains the most significant achievement in the history of the individual human being."5

"Ordinary algebra simply carries to a higher stage of usefulness in a special field the device which common language employs over the whole range of discourse."6

The Dependent Variable or Function

Just as an appreciation of change in varying magnitudes may contribute to the concept of an independent variable, so an appreciation of a correspondence between the elements of two changing quantities may lead to the concept of a dependent variable or a function of one variable. A knowledge of a dependent variable must rest upon a consciousness of correspondence; later it involves the idea of dependence.

The following definition of a function of one variable is due to E. H. Moore and is taken from the lectures of R. W. Barnard on the Theory of Functions of a Real Variable in the Summer Course at the University of Chicago, 1927.

1. Let p be a variable whose range is a class of objects P .
2. Let q be a second variable whose range is a second class of objects Q .
3. Let the variable q be so related to the variable p that when p is given a definite place in the range P , q has a definite

1T. P. Nunn, op. cit., p. 7.

2Ibid.

3Ibid., p. 8.

4Ibid., p. 9.

5Ibid., p. 7.

6Ibid.

value on the range Q ; then q is said to be a function of p .

4. The functional value q corresponding to p is expressed by the notation $q = f(p)$.

It is apparent that there are four essentials to the concept of dependent variable or function.

1. A class of objects P each of which may be designated by the independent variable p .

2. A class of objects Q each of which may be designated by the dependent variable q .

3. A form of correspondence, not necessarily one to one, which characterizes a relation between p and q such that when p is given a specified value or place on the range P , q is assigned a definite value or place on the range Q .

4. A functional relation between the variable classes which is expressed symbolically by $q = f(p)$.

For example, when we state that "the fragrance of the flowers varies with the time of day," we have in mind a set of states of fragrance of flowers, a set of time intervals for the day, a definite correspondence between the elements of the two variables fragrance and time intervals, and a functional relation implied in the term "varies."

Thus the mental attitude toward a functional correspondence is quite complex. It involves two one-dimensional surveys of the classes of elements for the two variables, an appreciation of the correspondence existing for these sets of elements, a review of the relations between particular and corresponding pairs of elements in the two variable classes, and finally a generalized statement concerning the set of relations. This attitude of mind is a second essential in successful functional thinking; it demands an appreciation of correspondence, a detailed survey of a set of related variable classes, and a generalization concerning a set of relations expressed by words or symbols appropriate to the sphere of thought wherein functional thinking is practiced. When the historian makes the statement, "The King of England is a constitutional monarch," he holds in mind a set of elements pertaining to the variable "the King of England" and a set of elements descriptive of the variable "constitutional monarch," he appreciates a correspondence between particular pairs of elements of the two variable classes, he surveys the set of relations which obtains in the set of corresponding pairs of elements, and finally he describes each and every one of the set of relations by the words "The King of England is a constitutional monarch." Thus the historian engages in functional thinking in the field of history.

The Mathematical Correlate to the
General Concept of Function

When the varying magnitudes are measurable, the classes of objects of the preceding definition are replaced by their measures. The fundamentals in the student's attention are changed from quantities to measures of quantities; the functional correspondence is that of two classes of numbers; the mathematician no longer considers the origin of the classes of related numbers as measures of physical quantities; they become the real objects of attention. As stated in Chapter II, page 12, of this report, the term function originated with the mathematician in his study of related sets of numbers; later generalizations make it possible to include the more general class of objects. The following definition is usually attributed to Dirichlet.¹

1. Let a and b be two fixed numbers.
2. Let x be a variable which assumes in turn all values between a and b .
3. Let y be a second variable whose value depends upon the value assigned to x in such a manner that to every value of x between a and b there corresponds a definite value of y .

Then y is said to be a function of the variable x in the interval from a to b . It is apparent that there are three essentials to the concept of function: a class of numbers X , each of which may be designated by the independent variable x in a range from a to b ; a class of numbers Y each of which may be designated by the dependent variable y ; and a form of correspondence, by means of which, to each value assigned to x in the range from a to b there is a definite value for y . One should note carefully that there is no restriction of the methods by which the value of y is assigned to the values of x ; the correspondence may be described by general statements, by rule, by table, by graph, by formula function, by strip-film, by machine, and so forth. The essential factor in a functional relation is the correspondence. For illustration of this thought, see Chapter II, page 17, of this report.

Functional Thinking

When one engages in functional thinking, one has in mind a functional correspondence between the elements of two variable classes of objects and is simultaneously engaged in discerning and interpreting the nature of the relations which obtain between particular and corresponding pairs of elements for the two variable classes and in endeavoring to express the common factor in a set of recognized relations by means of standard and approved methods

¹Op. cit., p. 132.

of expressing relationships. The relationships are not necessarily mathematical but, generally speaking, the student's aim is to express all such relationships numerically. Thus the mind holds and examines certain experiences concerning changing quantities as it surveys a correspondence of pairs of elements from two related variable classes; it seeks to educe relations between particular and corresponding pairs of elements and to express by generalizations using both words and symbols for words, the common factors in a set of relations. The mind seems to analyze particular relations by a kind of empirical method. When a relation has been found which satisfies one pair of corresponding elements in a functional correspondence, it is applied to a second pair; if it is successful, the relation is carried forward to a third pair; and so on. Thus two phases of mental activity appear to be continually active: (1) having given a pair of corresponding elements in two variable classes, the mind endeavors to decipher the relation between them; and (2) having given one element of a pair of corresponding elements in a functional correspondence and a relation between them, the mind seeks to determine the correlate element of the correspondence.

Spearman's "Noegenetic" Principles of Cognition

Let us accept with Spearman the three "noegenetic" principles of cognition which he has formulated as fundamentally basic laws in Educational Psychology to govern "primarily and qualitatively all genuine knowing and all growth of cognitive content."¹

1. "Any lived experience tends to evoke immediately a knowing of its direct attributes and its experiences."²

2. "The presenting of any two or more characters tends to evoke immediately a knowing of relations between them."³

3. "The presenting of any character together with a relation tends to evoke immediately a knowing of the correlative character."⁴

If we examine that method of thinking and learning which we have described as Functional Thinking in the light of these principles of learning, we gain immediately a broader concept of its significance and application in learning. It is apparent that Functional Thinking provides several distinct opportunities for the exercise of the foregoing principles. At the outset the

¹C. Spearman, The Nature of Intelligence and the Principles of Cognition, p. 241. London: Macmillan Co., 1923.

²Ibid., p. 48.

³Ibid., p. 63.

⁴Ibid., p. 91.

student apprehends lived experiences with changing quantities; such experiences are the outcome of a train of stimuli originating in the changing states of a varying magnitude; these experiences are necessarily particular and discrete in the student's mind and thus constitute the fundamentals as the mind seeks to educe certain relations between them. Later the student perceives and examines his lived experiences with a correspondence between two or more variable classes. The fundamentals of attention are now discrete pairs of corresponding elements in a functional correspondence as the mind seeks to evoke relations between the particular pairs of corresponding elements. In this examination for relations all three principles of learning are employed. In a third stage of functional thinking the fundamentals in the student's attention consist of a class of particular relations which have been discovered in the study of particular pairs of corresponding elements of the functional correspondence. Here the student attends to the set of particular relations that the mind may educe common elements and discover the general law from its particular manifestations. Finally the fundamentals of attention for the student include a class of standard and approved methods for expressing relationships. In this phase of functional thinking the student has a set of methods for expressing relationships; he has a general relationship to express and he seeks, by the application of the second and third principles, that mode of representation best suited to his appreciated need.

In addition to these several opportunities for the application of the foregoing principles of learning, Functional Thinking provides a general setting or plan for concerted and continuous study. This plan appears to include a motive for learning, an interest in study, and an objective justifying the investigation. The whole method appears to constitute a unit of learning which commences with experience, develops with interest, and closes with joy resultant upon discovery.

Functional Thinking in Elementary Algebra

1. The Functional Correspondence.— Having given two related classes of corresponding numbers described by the independent variable and the dependent variable to find the numerical relation between them. The discovery of numerical relation for such empirical exercises requires an application of the Second Principle of Cognition.

Given Class of Numbers. Independent Variable	x . .	1	2	3	4 . .
Unknown Numerical Relation. Formula-Equation					
Given Class of Numbers. Dependent Variable	y . .	2	5	10	17 . .

This investigation advocates that the presentation of Elementary Algebra in the secondary schools should commence with the study of Functional Correspondences. Such exercises as that illustrated above provide opportunity for the study of particular manifestations of an algebraic relation and a subsequent generalization.

2. The Formula-Equation.— Having given a class of numbers in a functional correspondence represented by the independent variable and the numerical relation inherent to the correspondence described by a formula-equation to find the corresponding class of numbers for the dependent variable. This exercise requires the application of the Third Principle of Cognition.

Given Class of Numbers. Independent Variable x . .	1	2	3	4	..
Given Numerical Relation. Formula-Equation $y =$	x^2+1				
Unknown Class of Numbers. Dependent Variable y . .					

Many recent presentations of Elementary Algebra urge that the study of Elementary Algebra should commence with exercises with the Formula-Equation. Such a presentation assumes that pupils understand the generalized formula. This assumption does not appear to be valid.

3. The Conditional Equation.— Having given the class of numbers in a functional correspondence represented by the dependent variable and the numerical relation inherent to the functional correspondence described by a formula-equation to find the corresponding class of numbers for the independent variable. This exercise is the solution of an algebraic equation. The Third Principle of Cognition is fundamental.

Unknown Class of Numbers. Independent Variable x ..				..
Given Numerical Relation. Conditional Equation $y =$	x^2+1			
Given Class of Numbers. Dependent Variable y ..	2	5	10	17..

Earlier presentations of Elementary Algebra were constructed about the Conditional Equation and assumed that pupils understood the meaning of the functional correspondence and a formula-equation. The omission of these fundamental concepts interferes with real progress in this secondary school subject.

CHAPTER IV

FUNCTIONAL RELATIONS AND ELEMENTARY ALGEBRA FOR GRADES IX AND X

Specialists in mathematics have recommended that elementary mathematics should be unified about the concept of function and so encourage the pupil to study dependence and related magnitudes as they advance in mathematics. Some ideas concerning the function have been presented in Chapter II but we must keep in mind that the mathematician meant more than this. Functional thinking is a mode of thought and has wide application in learning apart from its tremendously important place in mathematical thought and development. All interested in education would emphasize this phase of mental activity.

This study is concerned with one application of this recommendation. Here it is attempted to prove that an approach to elementary algebra may be closely associated with an elementary study of related changing quantities and with the numerical laws inherent to them. I shall endeavor to prove that pupil and teacher, alike, will find greater scope for the development and exercise of the many activities associated in the learning of algebra when presented in this way than in the usual deductive emphasis upon strange symbols and abstract formulas characteristic of many introductory courses in elementary algebra.

Instead of constructing the curriculum in elementary algebra to contain so many perfected expressions of numerical relations, I shall advocate that we study more closely the needs of the pupils as they learn and incorporate in the curriculum ways and means of supplying these needs to the pupils. Let us assist the pupils to assemble their equipment for learning new truths and arrange that they add to this equipment that which is essential to the new movement. Let us question their mental attitudes and introduce some means of modifying these attitudes as they become acquainted with strange paths in new fields of inquiry.

This divides the emphasis between the subject matter of the curriculum and the needs of the learner as he acquires the truth of this matter. For such results, we must design a curriculum replete with thought compelling situations which invite the pupils to make worthy effort, to sense the value of the truth, to signify a willingness for instruction and guidance, and to appre-

ciate the truth when gained. Only by such methods may we have the truth of the great masters shine anew in the enthusiasm of youth.

Since this study is concerned with method as well as with subject matter, this chapter may be regarded as another definition of function, adapted to the needs of teachers who are earnestly striving to follow the recommendation that pupils engage in functional thinking as they gather their experience with number. This chapter presents a brief outline of the topics selected for this study and the general method employed in presenting the truths of these topics while emphasizing the study of dependence. A more detailed account of the actual lessons used for the teaching program may be gathered by reference to the lessons in Appendixes A and B.

The characteristic features of this curriculum of method and subject matter are:

1. Whole situation, presenting related varying magnitudes, the measures of related quantities and sets of related numbers are introduced for general discussion throughout the whole program. The pupils are encouraged to investigate these numerical relations that, by inductive principles of learning, they may proceed from an examination of particular manifestations of a general numerical law to the consideration of the law itself. With the difficulties experienced in a verbal description of such numerical relationships they appreciate the meaning and use of the convenient algebraic symbol and the importance of the priceless formula. For example:
 - a) The first ideas of related quantities, of sets of related measures of quantities and of the formula, as a means of describing numerical relationships, originate with the study of a common, ordinary box.
 - b) The graphical method of recording numerical relations is explained in reference to the material gathered and presented for class study by the members of the class.
 - c) Some of the best problems for illustrating the meaning and application of new principles in algebra are found in those designed by the pupils for class discussion.
2. Wide experience with particular arithmetical processes, particularly with those dealing with signed numbers, is provided throughout the course in algebra for Grades IX and X. Practically every new phase of generalization in algebraic thought has its origin in the consideration of related sets of numbers designed to illustrate the principle under study. Such exercises provide the pupils with opportunity to review their arithmetic and to make the necessary generalizations for their science of algebra.

TOPICS AND LESSONS
FALL TERM, SEPTEMBER TO DECEMBER, 1931

No.	Lessons and Topics	Contribution to the Variable and to Functional Thinking
<u>Topic I—Review of Arithmetic</u>		
1.	Quantities and Measurements	Organizes the students' ex-
2.	Types of Numbers	perience with number in
3.	Arithmetical Operations, Integers	preparation for the broader
4.	" " Vulgar Fractions	viewpoint in number.
5.	" " Decimal "	
<u>Topic II—Varying Magnitudes</u>		
1.	Varying Magnitudes Defined	Acquaints the student with
2.	Varying Magnitudes Measured	changing quantities, the
3.	Related Magnitudes	measure of changing quanti-
4.	Related Magnitudes (continued)	ties, the dependence of
5.	Illustrative Problems (sharing)	some changing quantities
6.	" " (mensuration)	on others.
<u>Topic III—Signed Numbers</u>		
1.	Origin of Signed Numbers	The measurement of varying
2.	Representation of Signed Numbers	magnitudes leads to signed
3.	Addition of Signed Numbers	numbers. The operations
4.	Subtraction of Signed Numbers	with these must obtain be-
5.	Multiplication of Signed Numbers	fore one can consider
6.	Division of Signed Numbers	general situations.
<u>Topic IV—The Variable</u>		
1.	The Variable Defined	An acquaintance with the
2.	The Variable Illustrated	variable is necessary to
3.	The Invention of Variables	an appreciation of the
4.	The Number Range	formula with which the
5.	Number Range and Variable	student is to build alge-
6.	Simple Problems	bra.
<u>Topic V—Operations with Variables</u>		
1.	Addition and Subtraction	This topic provides for
2.	Multiplication and Parentheses	training in manipulation
3.	Division	of variables.
<u>Topic VI—The Equation</u>		
1.	The Identity	The equation has its origin
2.	The Conditioned Equation	with the variable. All of
3.	The Origin of Equations	this topic aims to build
4.	Solution of Problems	up in the pupil an appre-
5.	Review of Equation	ciation of the variable.
6.	Review of Variable	

TOPICS AND LESSONS
WINTER TERM, JANUARY TO MARCH, 1932

No.	Lessons and Topics	Contribution to the Variable and to Functional Thinking
<u>Topic I—The Variable, The Equation</u>		
1.	The Variable. Applications	In this topic the students study varying magnitudes, the variable, the equation, conditional and the identity, the solution of conditional equations.
2.	Solution of Problems	
3.	Algebra and Arithmetic	
4.	The Invention of Problems	
5.	Report of Student Invention	
6.	Discussion of Student Problems	
7.	Solution of Problems	
8.	Review of Axioms	
<u>Topic II—The Variable, Formula</u>		
1.	Formula Defined, Illustrated	The formula originates in a functional correspondence as a relation between the variables.
2.	Exercises in Construction of Formula	
<u>Topic III—Varying Magnitudes</u>		
1.	Student Study, Varying Magnitudes	Some variables are not related by formulas.
2.	Report of Student Study	
<u>Topic IV—The Identity, Factoring</u>		
1.	Factors Introduced, Illustrated	These lessons increase the pupil's experience with the identity and are thus related to the study of variables in functional correspondences.
2.	Polynomials with Monomial Factors	
3.	Illustrative Problems	
4.	Multiplication of Binomials	
5.	Factoring Trinomials	
6.	Multiplication of Binomials	
7.	Factoring Trinomials	
8.	Trinomial Squares	
9.	Difference of two Squares	
10.	Review of Factoring	
11.	General Review	
12.	Testing Program	

TOPICS AND LESSONS
SPRING TERM, APRIL TO JUNE, 1932

No.	Lessons and Topics	Contribution to the Variable and to Functional Thinking
<u>Topic I—The Graph</u>		
1.	The Graph of Statistical Data	The graph is one method of recording the measures of varying magnitudes. The bar graph is related to arithmetic as the continuous graph is related to algebra.
2.	The Pictograph as Introductory to the Graph	
3.	The Bar Graph of Linear Variables	
4.	The Continuous Graph of Linear Variables	
5.	The Bar Graph of Quadratic Variables	
6.	The Continuous Graph of Quadratic Variables	
7.	Bar Graph of Algebraic Polynomials	
8.	Continuous Graph of Algebraic Polynomials	
<u>Topic II—Review of Fundamental Operations</u>		
1.	Addition of Variables	In this review of the operations with the variable, the student is provided an opportunity in which to improve his ideas concerning the variables.
2.	Subtraction of Variables	
3.	Parentheses of Variables	
4.	Multiplication of Variables	
5.	Division of Variables	
<u>Topic III—General Review</u>		
1.	Type Products and Formulas	Review of the variable expressed by: Word, Table, Formula, Graph, Equation, Strip film.
2.	Factoring	
3.	Problems Involving Variables	
4.	Solution of Problems Designed by Students	
5.	Varying Quantities	
6.	Graphs of Related Varying Quantities	
7.	The Variable and the Formula	
8.	Testing Program June 3-21, 1932	

1. The Bar Graph of a Record of Events

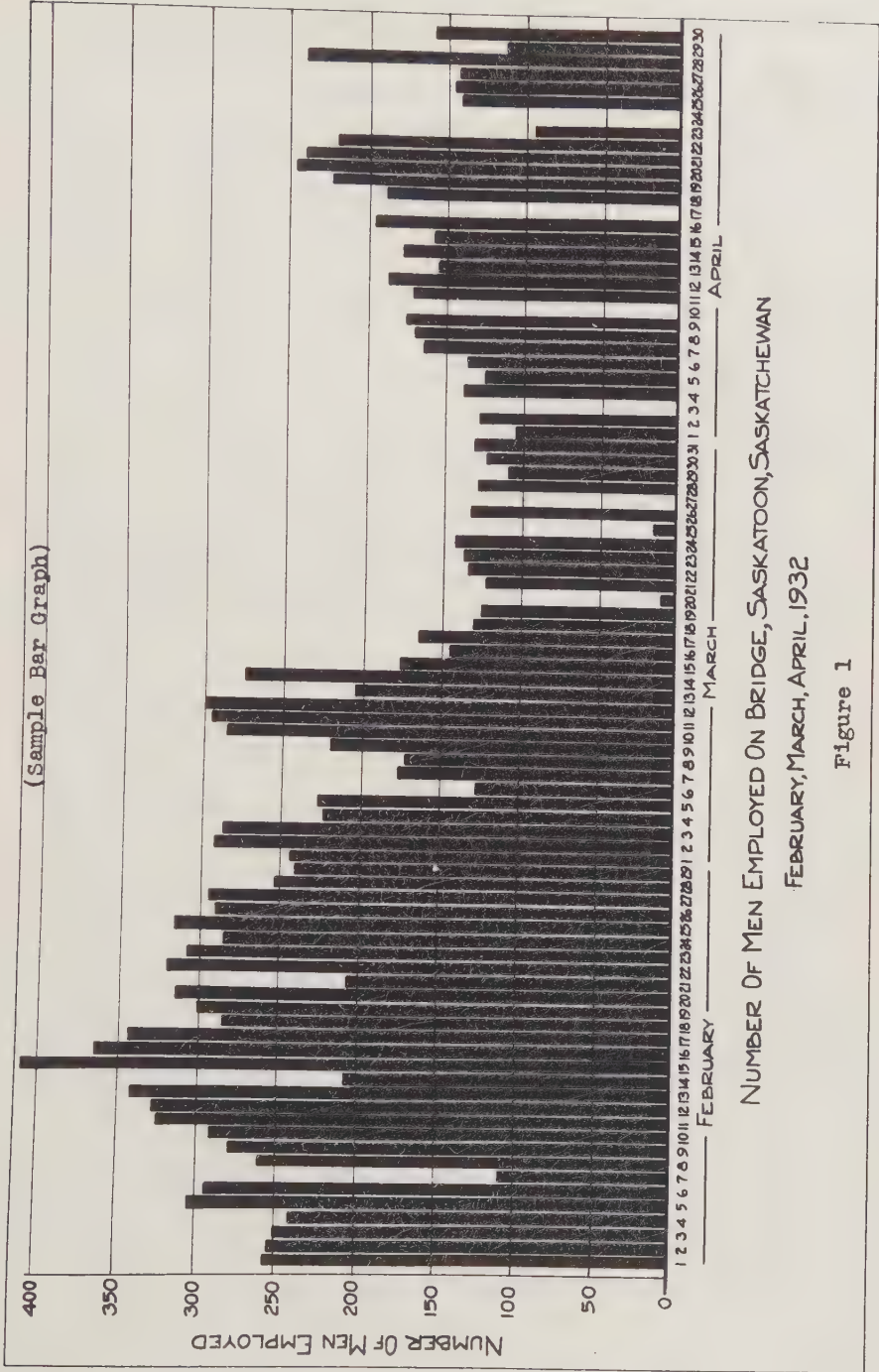
The accompanying bar graph pictures the number of men employed daily on the new bridge at Saskatoon during the months of February, March, and April, 1932. Study the graph carefully and answer the following questions:

- a) How many men were employed on March 17?
- b) On what days of this period were approximately 200 men employed?
- c) When this building project was undertaken, the people of Saskatoon expected it would give steady employment to its employees. Do you consider, after an inspection of this record, that the employment was steady throughout the three months under observation?
- d) Name three reasons which tend to make steady employment on such a building project exceedingly difficult.
- e) Mention one occupation in Saskatoon in which the employees do find steady employment.
- f) During what month were most men employed?
- g) During what month was the employment most steady?
- h) What was the average number of men employed during the week commencing April 10, 1932, and ending April 16, 1932?
- i) Point out one feature of this employment scheme that is well pictured by the graph.
- j) Which do you consider tells its story in the better manner—a table of corresponding values or a graph of corresponding values? Give reasons for your answer.

TABLE SHOWING NUMBER OF MEN EMPLOYED ON THE BRIDGE
AT SASKATOON FROM FEBRUARY 1-APRIL 30, 1932*

Date	February	March	April	Date	February	March	April
1	256	239	102	17	343	156	
2	253	288	125	18	279	131	186
3	245	284	25	19	300	125	219
4	239	216	132	20	316	6	240
5	305	219	123	21	204	124	235
6	288	125	133	22	320	127	214
7	119	173	162	23	309	130	93
8	259	170	164	24	282	132	
9	273	216	170	25	322	14	189
10	291	284		26	287	130	194
11	327	295	169	27	289		190
12	330	292	184	28	252	125	284
13	341	203	154	29	241	109	111
14	203	269	174	30		120	156
15	412	173	158	31		128	
16	363	146	190				

*Used at close of Examination Period for comparison.



2. The Variable

- a) (1) Explain what is meant by a variable quantity in a discussion.
(2) Explain what is meant by a variable.
(3) What are the two kinds of variables? Illustrate your answer with reference to the box problem (page 91).
- b) Name three methods which may be employed to exhibit the measurements of corresponding values of related varying magnitudes.
- c) If any length of the cut in the box problems be denoted by C invent dependent variables to describe:
(1) the length of any box.
(2) the area of the base of any box.
(3) the volume of any box.
- d) Consider the following table of corresponding values and answer the following questions:

	Length of Box	Width of Box	Area of Base of Box	Height of Box	Volume of Cylindrical Cake per Box
0	20	20	400	0	0
1	18	38	324	1	254 $\frac{4}{7}$
2	16	36	256	2	402 $\frac{2}{7}$
3	14	34	196	3	462
4	12	32	144	4	452 $\frac{4}{7}$
5	10	30	100	5	392 $\frac{6}{7}$
6	8	28	64	6	301 $\frac{5}{7}$
7	6	26	36	7	198
8	4	24	16	8	100 $\frac{4}{7}$
9	2	22	4	9	28 $\frac{2}{7}$
10	0	20	0	10	0

- (1) As the length of the cut increases from 0 to 10 units of length the width of the box increases from __ to __
decreases from __ to __
- (2) The girth of the box increases from __ to __
decreases from __ to __
- e) Examine this table of corresponding values for the related varying magnitudes and answer the following questions. The length of the cut increases from 0 to 10.

- (1) The length of the box (increases, decreases).
- (2) The width of the box (increases, decreases).
- (3) The height of the box (increases, decreases).
- (4) The perimeter of the base of box (increases, decreases).
- (5) The girth of box (increases, decreases).
- (6) The area of base of box (increases, decreases).
- (7) The total area of sides (increases, decreases).
- (8) The total area of base of box (increases, decreases).
- (9) The total area of the paper used in making the box (increases, decreases).
- (10) Volume of the box (increases, decreases).

f) With the table of values before you, plot the bar graphs of the following pairs of variables:

Length of cut	Length of cut	Length of cut	Length of cut
Length of box	Width of box	Height of box	Perimeter of base
Length of cut	Length of cut	Length of cut	Length of cut
Girth of box	Area of base	Area of sides	Volume of box

g) Let x denote the length of any cut and invent variables to describe the corresponding value of the following varying magnitudes:

- (1) The length of the box.
- (2) The width of the box.
- (3) The height of the box.
- (4) The perimeter of base of box.
- (5) The girth of box.
- (6) The area of base of box.
- (7) The area of sides of box.
- (8) The total area of paper used in making box.
- (9) The area of the paper wasted when removing the corners.
- (10) The volume of box.

h) Make a list of five formulas that you have used in your studies at this Collegiate Institute in subjects other than algebra.

i) The formula $V = c(20 - 2c)^2$ was illustrated by strip film.

$$V = c(20 - 2c)^2 = c(400 - 80c + 4c^2) = 400c - 80c^2 + 4c^3$$

$$V = \text{Volume of Box } [20 \times 20 \times c] - \text{Volume of 4 Boxes } [20 \times c \times c] + \text{Volume of 4 Boxes } [c \times c \times c].$$

The photograph of film used to illustrate the formula $V = 400c - 80c^2 + 4c^3$ follows.

ILLUSTRATING
THE
SIGNIFICANCE
OF THE
FORMULA

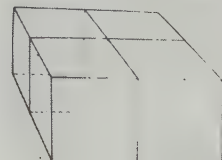
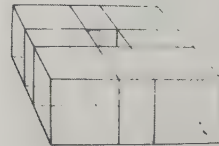
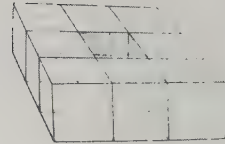
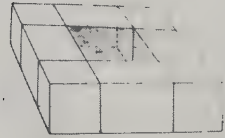
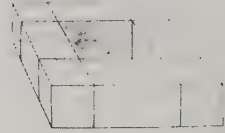
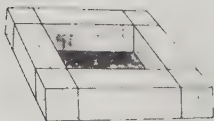
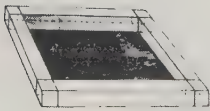
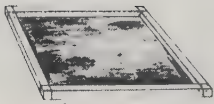


Figure 4

GRADE X, TOPICS AND LESSONS
FIRST TERM, SEPTEMBER TO DECEMBER, 1932

No.	Lessons and Topics	Nature of Contribution to the Study of Dependence
<u>Topic I—Linear Simultaneous Equations</u>		
1.	The variable reviewed	This topic provides opportunity for the review and extension of: Varying magnitudes; Dependent magnitudes; Methods of expressing a functional relation by words, table, symbol, and graph.
2.	Two independent variables	
3.	Origin of simultaneous equations	
4.	Solution of " "	
5.	Formula for solution of two simultaneous equations	There are two or more independent variables to be considered in this extension.
6.	Analysis of problems	
7.	Solutions of typical problems	For each condition relating the two independent variables we have either one of these variables dependent upon the other.
8.	Problem with <u>three</u> conditional equations	
9.	Indeterminate equations discussed	For the solution of a set of linear simultaneous equations there must be as many conditions as there are independent variables.
10.	Review analysis of problem with two simultaneous equations	
11.	Rapid analyses of ten easy problems with simultaneous equations	There are several ways of solving a pair of simultaneous equations, tabular, graphical, formula.
12.	Rapid analyses of ten more difficult problems	
13.	Review—origin of problems with simultaneous equations	The formula for solving simultaneous equations by determinants was studied.
14.	Tabular and graphical solution of two sets of simultaneous equations	
15.	Graphical solution of problems from Text	Tabular and graphical methods of solving equations are reviewed and extended.
16.	Review—methods of solving simultaneous equations	
17.	Indeterminate simultaneous linear equations discussed	Intersection of graphs introduced.
18.	General review of term's work	
<u>Topic II—Tests</u>		
19.	Test 1. Simultaneous equations	
20.	Test 2. Solution of problems	
21.	Test 3. Solution by tabular methods	

GRADE X, TOPICS AND LESSONS
SECOND TERM, JANUARY TO APRIL, 1933

No.	Lessons and Topics	Nature of Contribution to the Study of Dependence
<u>Topic I—Factoring, Algebraic Processes, and Exercises in Functional Thinking</u>		
1.	Review of elementary factor types	Whenever opportunity occurred, the presentation of the different types of factorable algebraic expressions is illustrated by a range of numbers.
2.	Factoring by grouping	
3.	Factors common to two expressions	
4.	Multiplication of binomials	The study of dependence is continually emphasized as the several reviews suggest.
5.	To form the square of algebraic expressions	
6.	Recognition of squared algebraic expressions	The formula is used continually to describe the factor types.
7.	Review. Dependence expressed by verbal statements	
8.	Difference of two squared terms	Discussion of factor types leads quickly to the discussion of identities or identical equations.
9.	Review. Dependence expressed by the formula	
10.	The formula is under review	The formula is used to describe an intricate number relation more easily than by verbal statement. It is a form of shorthand with its appropriate symbols, either variables or constants.
11.	Recognition of the difference of two squared forms	
12.	Review. Dependence expressed by tabular correspondence	
13.	"Incomplete square" types of algebraic expressions	
14.	Sum and difference of two cubed expressions and the formula for factorization	
15.	Review. Types of factoring	
16.	Test on work of term to date	
17.	Dependence expressed by ratio and scale	
18.	Dependence expressed by identities. Factor forms	
19.	Discussion of pupil difficulties	
20.	Tests. Recognition of factor types and factoring	
21.	Algebraic polynomials—meaning and form	

GRADE X, TOPICS AND LESSONS
THIRD TERM, APRIL TO JUNE, 1933

No.	Lessons and Topics	Nature of Contribution to the Study of Dependence
<u>Topic I--The Quadratic Equation</u>		
1.	Solution of a quadratic equation by factoring	The distinction between a formula and a conditioned equation is reviewed and strengthened.
2.	Solution of more difficult equations	
3.	Various methods of solving the quadratic equation	
4.	The solution of the general quadratic equation	
<u>Topic II--Ratio, Proportion, Variation</u>		
5.	Exercises requiring the consideration of a ratio	The pupils consider related quantities. The numerical relation is expressed by means of a ratio, by proportion, by variation, symbols, and by fractions.
6.	Exercises. Dependence expressed by ratio	
7.	Exercises. Variation, direct and inverse	
8.	Exercises. Dependence expressed by variation	
<u>Topic III--Fractions</u>		
9.	Simple exercises in fractions	Dependence expressed by formulas expressed in the form of algebraic functional expressions
10.	Addition and subtraction of fractions	
11.	Multiplication of fractions	
12.	Division of fractions	
<u>Topic IV--General Review</u>		
13.	Problems involving ratio and fractions	General review: dependence, correspondence, formula, equation.
14.	Problems reviewing dependence	
15.	Review--ratio, proportion, variation	
16.	A distinction between arithmetic and algebra	
17.	Dependence	
18.	Review	

CHAPTER V

TESTING PROGRAM AND EVIDENCE

This chapter presents an account of the testing program by which objective evidence for the demonstration of this thesis is to be secured. Its three parts describe, in turn, the several classes used in this study, the tests administered to these classes, and the scores for the several tests.

As already stated, the lessons outlined in Appendix A of this report have been taught to a class of pupils known throughout this investigation as the experimental class and designated by the symbol XB. These pupils attended the Nutana Collegiate Institute, Saskatoon, Saskatchewan, Canada, and were taught the lessons for Grade IX during 1931-32, and for Grade X in 1932-33. They continued their studies throughout Grade XI in 1933-34 and in June, 1934, wrote the Departmental Examination for the Second Class Teachers Diploma for the Province of Saskatchewan and Junior Matriculation to the Canadian Universities. The Principal of the Nutana Collegiate Institute during this period was Mr. A. W. Cameron, B.A.

Six control classes were employed in this study. Three of these, XC, XIB, and XIIC, were in attendance at the Nutana Collegiate Institute during 1932-33, taking the regular program of studies for Grades X, XI, and XII, as prescribed by the Regulations for the Province of Saskatchewan. The classes known as XD and XF were regular Grade X classes in attendance at the Central Collegiate Institute, Regina, Saskatchewan, under the principalship of Mr. W. G. Scrimgeour, M. A. Class XIIU designates a group of students in their first year at the University of Saskatchewan, Saskatoon, Saskatchewan, Canada, who, along with other subjects, were studying in 1932-33 an elementary course in Mathematical Analysis with Griffin, Mathematical Analysis, Volume I, as their text.

Mr. M. J. Sexsmith, B. A., and Mr. C. E. Morgan, B. A., were the regular mathematical masters at the Nutana Collegiate Institute and Mr. J. A. Cooper, B. A., was the teacher at the Regina Secondary School. These teachers are university graduates, specialists in mathematics, much interested in the presentation of mathematics in the secondary schools, and generally regarded by the Department of Education for the Province of Saskatchewan as

able and successful teachers.

Table XII gives a brief description of the various control classes and a ready comparison of the experimental and control units in chronological age (June, 1933) and mental age (December, 1932). The mean score on the Grade VIII Departmental Examination for the Province of Saskatchewan in June, 1931, is given for the four Grade X classes together with the mean score for the arithmetic of that examination.

The Terman Group Test of Mental Ability was administered to all pupils in December, 1932, under the direction of Professor S. R. Laycock, M. A., Ph. D., of the University of Saskatchewan. The mean mental age for the various groups has been computed from their scores on the Terman Test and included in the chart. The mental age for XD is 17 years, 2 months; for XF is 16 years, 4 months; if these classes are combined as a single control unit XD.F, the mean mental age is 16 years, 9 months. This result is slightly higher than the corresponding mean age for XB which is 16 years, 7 months. In the series of comparison records which follow in the later part of this chapter, XC, XD, and XF have been grouped to form a single control unit of Grade X pupils with a mean mental age of 16 years, 8 months. This group is designated by XC.D.F. In that this age is quite close to that for XB we compare the responses to the various tests for two groups of pupils of approximately equal mental ability.

At the close of their elementary school studies, most pupils write the Grade VIII Examination for Entrance to the Secondary School. This examination is conducted each year by the Department of Education for the Province of Saskatchewan. Mean scores for this examination and for that part testing the pupils in arithmetic have been computed and are included that we may regard these examinations as inventory tests for this investigation. These scores provide additional evidence that there is no significant difference in either general ability or arithmetical efficiency for the four Grade X classes. It will be noted that XD is the youngest class and has the highest mental age.

As stated, the Grade X classes were following the regular Program of Studies as described in the Regulations issued each year by the Department of Education. This program includes a course in Elementary Algebra to be given throughout the three years in Grades IX, X, and XI; hence the four classes, XB, XC, XD, and XF, are taught the same topics throughout the two years of this study. The topics in elementary algebra, as outlined in the Regulations for the Province of Saskatchewan and authorized by the Department of Education for this Province, were chosen as appropriate to this study, in that this study is more concerned with

TABLE XII
COMPARISON OF CONTROL CLASSES WITH EXPERIMENTAL CLASS*

Class and School	Mean Chronological Age June, 1933	Mean Terman Score Dec., 1932	Mean Mental Age	Mean Score and Standard Deviation of Grade VIII, June, 1931; Grade VIII Arithmetic
XB - SNCI	16y 4m $\pm$ 0.92m σ = 7.89m $\pm$ 0.65m	147.55 $\pm$ 1.84 σ = 16.01 $\pm$ 1.30	16y 7m	627.27 $\pm$ 9.42 σ = 81.85 $\pm$ 6.65
XC - SNCI	16y 2m $\pm$ 1.29m σ = 10.35m $\pm$ 0.88m	150.50 $\pm$ 2.88 σ = 24.01 $\pm$ 2.04	16y 8m	610.00 $\pm$ 8.40 σ = 66.65 $\pm$ 5.93
XD - RCCI	15y 9m $\pm$ 1.40m σ = 13.14m $\pm$ 0.99m	160.35 $\pm$ 1.96 σ = 18.31 $\pm$ 1.39	17y 2m	584 $\pm$ 16.92 σ = 103.97 $\pm$ 11.95
XE - RCCI	16y 4m $\pm$ 1.34m σ = 11.80m $\pm$ 0.95m	143.65 $\pm$ 2.39 σ = 19.65 $\pm$ 1.57	16y 4m	79.40 $\pm$ 2.33 σ = 14.34 $\pm$ 1.65
XIB - SNCI	17y 3m $\pm$ 1.38m σ = 12.19m $\pm$ 0.98m	167.00 $\pm$ 2.63 σ = 23.75 $\pm$ 1.86	17y 6m	
XIIC - SNCI	18y 3m $\pm$ 1.31m σ = 11.69m $\pm$ 0.92m	175.80 $\pm$ 2.08 σ = 19.01 $\pm$ 1.47	17y 10m	
XIIU - U. of S.	21y	162.10 $\pm$ 2.97 σ = 23.37 $\pm$ 2.10	17y 4m	

* SNCI - Nutana Collegiate Institute, Saskatoon, Saskatchewan, Canada;
RCCI - Central Collegiate Institute, Regina, Saskatchewan, Canada;
U. of S. - University of Saskatchewan, Saskatoon, Saskatchewan, Canada.

the encouragement of Functional Thinking in elementary algebra than with the selection of topics. At the close of this experimental study in Grades IX and X, the pupils were prepared to continue with their elementary algebra in Grade XI without any serious disturbance due to interference with their preparatory work in Grades IX and X.

For all classes, three periods of approximately forty minutes each per week were allocated for the teaching and study of algebra. This means that there is only one essential difference in the presentation of elementary algebra to these four classes and this is due to the emphasis upon the variable for the experimental class. XB is taught its algebra by means of the lessons outlined in Appendixes A and B, while XC, XD, and XF are taught in the usual manner. A brief survey of the responses to the testing program reveals many differences; most of these are due to different methods employed in presenting the subject matter of the course.

Three tests are used in this study:

1. The Breslich Algebra Survey Test, First Semester, Form A, given in December, 1932, to all Grade X classes.
2. The Breslich Algebra Survey Test, Second Semester, Form A, administered to all Grade X classes in June, 1933.
3. A form of the Breslich Functional Thinking Test, as described in this report. This test was given to all control classes in June, 1933.

Comparison Record for Grade X Classes

The Breslich Algebra Survey Test, First Semester, Form A

Part I. Algebraic Concepts

1. The Test

This test is to show whether you know the meanings of some words that are frequently used in algebra. Each of the twelve algebraic expressions below illustrates one of the twelve words. The words are:

- | | | |
|---------------------|----------------------|-----------------------|
| (1) Radical | (5) Ratio | (9) Literal number |
| (2) Polynomial | (6) Trinomial square | (10) Binomial |
| (3) Negative number | (7) Linear equation | (11) Product |
| (4) Similar terms | (8) Dissimilar terms | (12) Linear trinomial |

You are to write in each square the number of the word which the algebraic expression illustrates. Thus, a/b illustrates the word "Ratio." Hence number 5 is written in the square. $a + b + c$ illustrates a "Linear trinomial." Hence 12 is written in the square.

(1) a/b	5	(7) -2	
(2) $a + b + c$	12	(8) $a - b$	
(3) $x + y + z - p + q$		(9) $3x, 4x, 10x$	
(4) $\sqrt{ab}$		(10) $3x, 4y, z$	
(5) $(x + 1)(x - 2)$		(11) $x + 5 = 3x + 4$	
(6) a		(12) $x^2 + 2xy + y^2$	

Part I Score = number right $\times 2 = 20$

2. Character of Test

The author presents ten algebraic expressions selected from elementary algebra and the ten corresponding words which describe these expressions. He requests the pupils to examine the expressions, to study their meaning, and to select from the list of words that one which best describes the expression under observation. The functional thinking involved is that inherent to forming a correspondence between algebraic expressions and algebraic terms descriptive of fundamental concepts.

3. Interpretation of Test and Score

- The pupils have been taught the material for this test with exception of the term "Radical."
- This exercise presents an entirely new situation to all pupils and is one which tests an exceedingly important phase of functional thinking.
- The score (Table XIV) for XB is higher than that for the other classes. The difference of the mean scores is significant for XF, XD.F, and XC.D.F.

TABLE XIII

ACCURACY AND SPEED - I. ALGEBRAIC CONCEPTS

Correspondence			XB			XC			XD			XF		
No.	Expression	Word	R	W	N	R	W	N	R	W	N	R	W	N*
3.	$x + y + z - p + q$	Polynomial	22	8	4	13	8	8	7	12	17	4	18	12
4.	$\sqrt{ab}$	Radical sign	4	6	24	1	9	19	5	8	23	2	11	21
5.	$(x + 1)(x - 2)$	Product	12	11	11	8	7	14	20	12	4	19	9	6
6.	a	Literal number	28	1	5	20	2	7	24	4	8	13	8	8
7.	-2	Negative number	33	..	1	26	1	2	33	1	2	30	3	1
8.	$a - b$	Binomial	13	11	10	10	4	15	11	11	14	7	11	16
9.	$3x, 4x, 10x$	Similar terms	23	3	8	16	1	12	20	1	13	12	8	14
10.	$3x, 4y, z$	Dissimilar terms	16	3	15	12	4	13	16	4	16	10	9	15
11.	$x + 5 = 3x + 4$	Linear equations	18	4	12	16	3	10	23	7	6	20	9	5
12.	$x^2 + 2xy + y^2$	Trinomial square	25	4	5	15	4	10	18	7	11	10	14	10
Total			194	51	95	137	43	110	117	67	116	132	100	108

*R = Right; W = Wrong; N = Not attempted.

TABLE XIV
COMPARISON SCORE - I. ALGEBRAIC CONCEPTS

Score	Class					
	XB	XC	XD	XF	XD.F	XC.D.F
20	.	.	1	1	2	2
18	2	.	.	.	.	.
16	4	.	1	.	1	5
14	6	4	4	.	4	9
12	7	5	6	.	9	13
10	4	4	12	3	22	23
8	7	1	3	10	8	13
6	4	5	6	5	11	15
4	.	4	2	9	11	15
2	.	4	1	1	2	3
0	.	1	.	.	.	1
Population	34	1	36	34	70	99
Mean Score	11.41 ± .41	9.45 ± .58	9.84 ± .40	7.76 ± .41	8.80 ± .30	9.00 ± .27
Standard Deviation	3.52 ± .29	4.64 ± .41	3.60 ± .29	3.54 ± .29	3.71 ± .21	4.02 ± .19
Difference Mean Score	.	1.96 ± .	1.57 ± .	3.65 ± .58	2.61 ± .50	2.41 ± .49
Coefficient of Reliability	.	1.96 ± .	1.57 ± .	3.65 ± .58	2.61 ± .50	2.41 ± .49
Probability of Significant Difference	.	.937	.935	1.00	1.00	1.00

TABLE XXXIII

SUMMARY - BRESLICH ALGEBRA SURVEY TEST - FIRST SEMESTER

Test		XB	XC	XD	XF	XD.F	XC.D.F
1.	Mean Score	11.41 $\pm$.407	9.45 $\pm$.581	9.84 $\pm$.405	7.76 $\pm$.409	8.80 $\pm$.299	9.00 $\pm$.272
	Standard Deviation	3.52 $\pm$.288	4.64 $\pm$.411	3.60 $\pm$.286	3.54 $\pm$.289	3.71 $\pm$.212	4.02 $\pm$.193
	Difference Mean						
	Score		1.96 $\pm$.709	1.57 $\pm$.574	3.65 $\pm$.576	2.61 $\pm$.505	2.41 $\pm$.489
	Coefficient of Reliability		1.96 $\pm$.709	1.57 $\pm$.574	3.65 $\pm$.576	2.61 $\pm$.505	2.41 $\pm$.489
2A.	Probability of Significant Difference						
	Mean Score		0.937	0.935	1.000	1.000	1.000
	Standard Deviation	6.38 $\pm$.214	7.40 $\pm$.232	7.47 $\pm$.129	7.91 $\pm$.135	7.68 $\pm$.096	7.60 $\pm$.096
	Difference Mean	1.86 $\pm$.151	1.85 $\pm$.164	1.15 $\pm$.091	1.17 $\pm$.096	1.19 $\pm$.068	1.41 $\pm$.068
	Score		1.02 $\pm$.315	1.09 $\pm$.250	1.53 $\pm$.253	1.30 $\pm$.234	1.22 $\pm$.234
2B.	Coefficient of Reliability		1.02 $\pm$.315	1.09 $\pm$.250	1.53 $\pm$.253	1.30 $\pm$.234	1.22 $\pm$.234
	Probability of Significant Difference						
	Mean Score		0.971	0.997	1.000	1.000	1.000
	Standard Deviation	7.15 $\pm$.208	7.55 $\pm$.204	7.75 $\pm$.152	8.26 $\pm$.158	8.06 $\pm$.111	7.87 $\pm$.100
	Difference Mean	1.80 $\pm$.147	1.63 $\pm$.144	1.36 $\pm$.108	1.37 $\pm$.112	1.38 $\pm$.079	1.47 $\pm$.070
	Score		0.40 $\pm$.291	0.60 $\pm$.258	1.11 $\pm$.261	0.85 $\pm$.235	0.72 $\pm$.231
	Coefficient of Reliability		0.40 $\pm$.291	0.60 $\pm$.258	1.11 $\pm$.261	0.85 $\pm$.235	0.72 $\pm$.231
	Probability of Significant Difference						
	Mean Score		0.644	0.885	0.996	0.985	0.966
	Standard Deviation						

TABLE XXXIII (Continued)

Test	XB	XC	XD	XF	XD.F	XC.D.F
3. Mean Score	12.60 ± .294	14.00 ± .415	14.00 ± .227	14.36 ± .310	14.14 ± .185	14.10 ± .182
Standard Deviation	2.54 ± .208	3.31 ± .293	2.02 ± .161	2.66 ± .220	2.30 ± .131	2.68 ± .128
Difference Mean						
Score		1.40 ± .508	1.40 ± .371	1.76 ± .427	1.54 ± .348	1.50 ± .345
Coefficient of Re-						
liability		1.40 = -2.75	1.40 = -3.76	1.76 = -4.12	1.54 = -4.14	1.50 = -4.35
Probability of		.508	.371	.427	.348	.345
Significant Dif-						
ference		0.936	0.988	0.995	0.996	0.997
4. Mean Score	12.53 ± .535	11.07 ± .586	10.75 ± .516	9.53 ± .371	10.16 ± .324	10.42 ± .287
Standard Deviation	4.52 ± .378	4.68 ± .414	4.59 ± .365	3.21 ± .263	4.02 ± .229	4.24 ± .203
Difference Mean						
Score		1.46 ± .793	1.78 ± .743	3.00 ± .651	2.37 ± .625	2.11 ± .607
Coefficient of Re-						
liability		1.46 = +1.84	1.78 = +2.40	3.00 = +4.61	2.37 = +3.80	2.11 = +3.47
Probability of		.793	.743	.651	.625	.607
Significant Dif-						
ference		0.785	0.895	0.998	0.990	0.982
5A. Mean Score	10.71 ± .363	13.10 ± .358	12.88 ± .359	11.94 ± .405	12.43 ± .272	12.63 ± .220
Standard Deviation	3.14 ± .257	2.86 ± .253	3.19 ± .254	3.50 ± .286	3.38 ± .193	3.25 ± .156
Difference Mean						
Score		2.39 ± .510	2.17 ± .512	1.23 ± .544	1.72 ± .454	1.92 ± .424
Coefficient of Re-						
liability		2.39 = -4.68	2.17 = -4.24	1.23 = -2.26	1.72 = -3.79	1.92 = -4.52
Probability of		.510	.512	.544	.454	.424
Significant Dif-						
ference		0.998	0.996	0.872	0.988	0.998
5B. Mean Score	3.32 ± .104	2.38 ± .165	3.05 ± .129	3.09 ± .130	3.07 ± .092	2.87 ± .084
Standard Deviation	0.90 ± .074	1.32 ± .117	1.15 ± .091	1.12 ± .092	1.14 ± .065	1.24 ± .059
Difference Mean						
Score		0.94 ± .195	0.27 ± .165	0.23 ± .165	0.25 ± .138	0.45 ± .134

TABLE XXXIII (Continued)

Test	XB	XC	XD	XF	XD.F	XC.D.F
Coefficient of Reliability		$\frac{0.94}{.195} = +4.82$	$\frac{0.27}{.165} = +1.63$	$\frac{0.23}{.165} = +1.40$	$\frac{0.25}{.138} = +1.81$	$\frac{0.45}{.134} = +3.36$
Probability of Significant Difference						
Mean Score	$2.41 \pm .134$	0.999	0.719	0.655	0.775	0.976
Standard Deviation	$1.16 \pm .095$	$1.14 \pm .139$	$1.72 \pm .126$	$1.47 \pm .123$	$1.60 \pm .089$	$1.46 \pm .076$
Difference Mean		$1.11 \pm .098$	$1.12 \pm .089$	$1.06 \pm .087$	$1.10 \pm .063$	$1.12 \pm .054$
Score		$1.27 \pm .193$	$0.69 \pm .184$	$0.94 \pm .182$	$0.81 \pm .161$	$0.95 \pm .154$
Coefficient of Reliability						
Probability of Significant Difference		$\frac{1.27}{.193} = +6.51$	$\frac{0.69}{.184} = +3.75$	$\frac{0.94}{.182} = +5.17$	$\frac{0.81}{.161} = +5.04$	$\frac{0.95}{.154} = +6.17$
Mean Score	$7.06 \pm .282$	1.000	0.988	1.000	1.000	1.000
Standard Deviation	$2.44 \pm .200$	$6.90 \pm .333$	$6.78 \pm .272$	$5.82 \pm .303$	$6.31 \pm .207$	$6.48 \pm .178$
Difference Mean		$2.66 \pm .236$	$2.42 \pm .192$	$2.62 \pm .214$	$2.57 \pm .147$	$2.62 \pm .125$
Score		$0.16 \pm .436$	$0.28 \pm .391$	$1.24 \pm .413$	$0.75 \pm .349$	$0.58 \pm .333$
Coefficient of Reliability						
Probability of Significant Difference		$\frac{0.16}{.436} = +0.37$	$\frac{0.28}{.391} = +0.72$	$\frac{1.24}{.413} = +3.00$	$\frac{0.75}{.349} = +2.14$	$\frac{0.58}{.333} = +1.74$
Mean Score		0.000	0.000	0.957	0.852	0.763
Standard Deviation	$1.24 \pm .160$	$2.14 \pm .228$	$0.46 \pm .098$	$0.30 \pm .110$	$0.38 \pm .070$	$0.90 \pm .100$
Difference Mean	$1.38 \pm .112$	$1.82 \pm .161$	$0.87 \pm .069$	$0.872 \pm .071$	$0.872 \pm .050$	$1.48 \pm .071$
Score		$0.90 \pm .278$	$0.78 \pm .186$	$0.94 \pm .188$	$0.86 \pm .174$	$0.34 \pm .188$
Coefficient of Reliability						
Probability of Significant Difference		$\frac{0.90}{.278} = -3.23$	$\frac{0.78}{.186} = +4.20$	$\frac{0.94}{.188} = +5.00$	$\frac{0.86}{.174} = +4.37$	$\frac{0.34}{.188} = +1.81$

TABLE XXXIII (Continued)

Test	XB	XC	XD	XF	XD.F	XC.D.F
Probability of Significant Difference		0.971	0.995	1.000	0.997	0.775

- Part I. Algebraic Concepts.
Part II. Algebraic Processes.
 A. Addition and Subtraction.
 B. Multiplication and Division.
Part III. Solving Equations.
Part IV. Deriving Equations.

Part V. Functional Relations.
 A. Evaluating Formulas.
 B. Deriving Formulas from Problems.
 C. Interpreting Graphs.
Part VI. Algebraic Expressions.
 A. Factoring.
 B. Fractions.

CHAPTER VI

TESTING PROGRAM AND EVIDENCE - BRESLICH FUNCTIONAL THINKING TEST

This chapter continues the account of the program of tests. It describes the Breslich Functional Thinking Test administered to the pupils of the experimental class, XB, and to those of the control classes in June, 1933. The Breslich Functional Thinking Test is divided into six parts:

1. Recognizing Relationships.
2. Recognizing Relationships in Unfamiliar Formulas.
3. How a Change in One Variable Affects the Others.
4. Interpreting Graphical Representation of Functional Relationships.
5. Expressing Functional Relationships Symbolically.
6. Variation and the Several Modes of Describing It.

The control classes for this program include the three Grade X classes employed with the Breslich Algebra Survey Test as described in Chapter V. In addition there are one Grade XI class and one Grade XII class attending the Nutana Collegiate Institute, Saskatoon, Saskatchewan, Canada, and one class in Elementary Mathematical Analysis from the University of Saskatchewan, 1932-33. These last three classes are described by the symbols XIB, XIIC, and XIIU respectively. The results of the tests indicate clearly that pupils in Grades X, XI, and XII have learned much about functional relationship without definite teaching concerning this subject but that the records for the three classes XB, XIIC, and XIIU are decidedly improved by the increased attention given to the study of functional relations and their representation by verbal statement, tabular correspondence, graph, and symbol. A full description of the test and score follows.

Comparison Record for All Classes - Breslich Functional Thinking Test Part I. Recognizing Relationships A. Dependence in Life Situations

1. The Test

The number of movie tickets that can be bought with a dollar bill depends on the price of each ticket. The cost of a steak depends on the size of the steak and the price per

pound. Thus, one number often depends on another. To show how well you recognize such dependence write on each blank line the word, or words, completing the sentence.

- (1) The cost of a 50-pound bag of sugar depends on the _____ per pound.
- (2) Barring accidents and delays, the number of hours that it will take an automobile to travel 150 miles depends on the _____
- (3) The number of pencils that can be purchased for a quarter depends on the _____ of one pencil.
- (4) The amount of pay a man receives depends on the number of _____ he works and his _____ per day.

Score for Part I, Sec. A, = Number right = 4

2. Character of the Test

The author presents four incomplete statements concerning dependence in life situations and requests the pupils to complete these statements. Thus the author tests the pupils in the recognition and appreciation of relationships. The functional thinking required in this exercise is that incident to an appreciation of the fact that many magnitudes depend for value upon the value of a second independent magnitude. For example, the temperature of a room depends upon the outside temperature.

3. Interpretation of Test and Score

- a) Most pupils appreciate dependence in life situations.
- b) The difference in mean scores (Table LIX) for this test contributes little of value for this study although it agrees with the findings of Knight.¹

¹T. M. Knight, "Progressive Development of Functional Thinking in Mathematics." Unpublished Master's thesis, Dept. of Education, University of Chicago, 1933.

TABLE IIX
COMPARISON SCORE - I A. DEPENDENCE IN LIFE SITUATIONS

Score	Class									
	XB	XIIU	XIIC	XIB	XC	XD	XF	XD.F	XC.D.F	
4	36	44	38	35	33	42	37	79	112	
3		1		2						
2										
1										
0										
Population	36	44	38	35	33	42	37	79	112	
Mean Score	4.00 ± .00	3.98 ± .01	4.00 ± .00	3.95 ± .02	4.00 ± .00	4.00 ± .00	4.00 ± .00	4.00 ± .00	4.00 ± .00	
Standard Deviation	0.00 ± .00	.148 ± .01	0.00 ± .00	.226 ± .02	0.00 ± .00	0.00 ± .00	0.00 ± .00	0.00 ± .00	0.00 ± .00	
Difference Mean		.02 ± .01	0.00	0.05 ± .02	0.00	0.00	0.00	0.00	0.00	
Coefficient of Reliability		.02 .01		.05 .02	0.00	0.00	0.00	0.00	0.00	
Probability of Significant Difference		±2.0		±2.8						
		.823		.941						

CHAPTER VII

ORGANIZATION OF EVIDENCE

This chapter presents some evidence of success for the modified presentation of elementary algebra to the pupils of the experimental class XB when based upon the study of functional relations. This evidence has been organized into four sections, which follow.

I. The scores from the Breslich Algebra Survey Tests have been combined and arranged in five divisions: Algebraic Concepts, Algebraic Processes, Arithmetical Processes, Appreciation of Dependence, and Tabular and Graphical Records of Functional Relations. The success of the various classes has been computed in these phases of algebra and comparison records presented for examination.

II. The scores from the Breslich Algebra Survey Tests have been so arranged that one may compare the progress and success of pupils of three levels of mental ability: those above the upper quartile measure, those in the inter quartile division, and those below the lower quartile measure for pupils of the experimental class XB and those from the combined control class XC.D.F.

III. The scores from the Breslich Functional Thinking Test have been rearranged into five divisions: Recognition of Relationships, Related Changes in the Variables, Graphical Studies, Expression of Relationships by Symbols, and Variation. Statements of the success of the various control classes in these several phases of functional thinking have been prepared and submitted for comparison.

IV. Comparison Records for several sets of pupils' answers to certain exercises from the Breslich Functional Thinking Test have been selected and submitted as additional evidence of success of a particular and special nature.

I. Classification of Evidence and Inferences from Scores from the Breslich Algebra Survey Tests

As stated, this section contains the results of the examination with the two algebra tests grouped together in five topics: Algebraic Concepts, Algebraic Processes, Solution of Equations, Appreciation of Dependences, and Graphical Records, that the suc-

cess of the various classes on these topics might be considered separately and compared.

A. Scores and Inferences from the Breslich
Algebra Survey Tests—Algebraic Concepts

1. The Comparison Record in Table LXXXV indicates that the experimental class XB has a distinct gain over the control classes in the combined tests on Algebraic Concepts.

2. All tests in Table LXXXVI show a superior response for the experimental class. The inference is that the modified presentation of elementary algebra makes wider provision for the presentation of these fundamental concepts and that XB has a better grasp of them.

3. From Table LXXXVI we learn that XB's record is better for the second test than for the first, and this suggests that XB is gaining in its appreciation of the fundamental concepts as it advances in elementary algebra.

4. As already stated in this report, page 162, the pupils were disturbed by the unfamiliar form in which the division problem is set up and this interfered with their answers to Test IB, Second Semester.

5. Throughout this presentation of elementary algebra to Grades IX and X, careful and continuous attention has been directed toward the meaning and use of algebra. The pupils consider arithmetical situations toward which they are encouraged to bring algebraic attitudes. Later they introduce and design appropriate symbols with which to describe certain numerical laws. Operations with the symbols are delayed. Thus, it is hoped, the pupils will work into their algebra rather than learn about finished forms from the completed textbook. As a result of this shift of emphasis, much greater attention is available for the new terms and strange symbols as found in elementary algebra. The foregoing response suggests that such a presentation is valid.

6. When pupils learn from such textbooks as that authorized for this class, many forms are presented without adequate explanation. The ideas behind these forms must be discovered by the pupils or supplied by the teacher. The results of the foregoing test indicate that the curriculum would be improved if more opportunities were provided wherein the pupils were guided to appreciate the significance, appropriateness, and beauty of new terms and expressions as they learn them. Without these opportunities, generalizations are accepted as facts, there is no substantial experience behind them, and they are soon forgotten.

TABLE LXXXV
COMPARISON RECORD - ALGEBRAIC CONCEPTS

Total Score 36.50	XB	XC	XD	XF
Population .	31	29	35	33
Mean Score .	24.49 $\pm$ 0.51	19.88 $\pm$ 0.79	19.40 $\pm$ 0.39	16.70 $\pm$ 0.42
Standard Deviation .	4.21 $\pm$ 0.36	6.29 $\pm$ 0.56	3.54 $\pm$ 0.28	3.99 $\pm$ 0.33
Difference Mean Score		4.61 $\pm$ 0.94	5.09 $\pm$ 0.64	7.79 $\pm$ 0.69
Coefficient of Relia- bility		$\frac{4.61}{.94} = +5.00$	$\frac{5.09}{.64} = +7.76$	$\frac{7.79}{.69} = +11.26$
Probability of Signifi- cant Dif- ference		1.00	1.00	1.00

TABLE LXXXVI
COEFFICIENTS OF RELIABILITY FOR DIFFERENCES OF MEAN SCORES

Description of Test	Semester	XC	XD	XF
1. Algebraic Concepts	First, Applied Dec., 1932	+2.76	+ 2.74	+ 6.33
1A. Algebraic Concepts	Second, Applied June, 1933	+6.18	+13.97	+13.12
1B. Algebraic Concepts	Second, Applied June, 1933	+2.57	- 0.08	+ 1.07

CHAPTER VIII

SUMMARY OF DIFFERENCES IN MATHEMATICAL THOUGHT AND ABILITY BETWEEN THE PUPILS OF THE EXPERIMENTAL CLASS XB AND THE PUPILS OF THE GRADE X CON- TROL CLASSES AS REVEALED BY THEIR RESPONSES TO THE SEVERAL STANDARD TESTS

The experimental class XB has graduated from its two years of training in elementary algebra, which was organized and presented about the broad concepts of variable and function. The class has since proceeded, with the other Grade X classes, to its studies in Grade XI. In June, 1934, these pupils wrote the Departmental Examination for a Teachers Diploma for the Province of Saskatchewan and for Junior Matriculation to the Canadian Universities with the following result in Grade XI Elementary Algebra:

	Class			
	XB	XC	XD	XF
Mean Mental Age	16 y, 6 m	16 y, 8 m	17 y, 1 m	16 y, 3 m
Mean Algebraic Score . .	83.9	84.4	88.3	82.5

In its responses to the program of testing with three standard tests and six control classes, the experimental class has exhibited several distinct and characteristic tendencies in mathematical thought and ability, which are presented in this summary. This chapter points out wherein XB differs from the Grade X control classes and presents a review of the objective evidence from the testing program which justifies the conclusions concerning the difference in the several classes. The following statements suggest the most important phases of the difference between the experimental and control classes at the close of this teaching and testing program.

1. The pupils of the experimental class possess a broader viewpoint in elementary algebra.
2. They manipulate the symbols in algebra and arithmetic rather less skilfully.
3. They recognize and estimate change in quantities.

4. They appreciate dependence of one quantity upon another.
5. They understand elementary functional correspondences.
6. They are making healthy progress in elementary algebra at the close of Grade X.
7. They engage readily in functional thinking.
8. They excel in the fundamental concepts of elementary algebra.
9. They analyze rather better than they generalize in elementary algebra.
10. They possess an algebra which promises to be of great social value.

A Broader View of Algebra

The pupils of XB possess a broader concept of elementary algebra than that of the other Grade X classes. Their answers to the various tests indicate that these pupils appreciate algebraic situations when presented in verbal statements, in tabular correspondences, in graphical records, in formulas, and in algebraic expressions. The usual more specialized study of formulas by means of the generalized algebraic expressions was postponed until a body of experience with number relations and a wider view of the reasons for algebra had been established with the pupils. The results from the set of tests shown in Table CIX, selected from those employed in this study, are submitted as objective evidence for the foregoing conclusions.

The record (Table CIX) does not reveal that XB has always a high score but it does indicate that the experimental class has in general a superior record for this set of tests, hence the conclusion concerning the possession of a broader concept of the meaning and use of elementary algebra.

Manipulation of Symbols in Algebra and Arithmetic

The responses from the pupils of the experimental class to those exercises which tested them in accuracy and speed in operations with signed numbers in arithmetic and symbols in algebra indicate that they were not as strong with these phases of elementary algebra as were the pupils in the control classes. Further study of the accompanying Accuracy and Speed Report reveals that the pupils from XB were slow with their responses rather than inaccurate. This apparent weakness should be regarded as peculiar to this particular method of presenting elementary algebra rather than as a weakness of the general principle of combining elementary algebra with the variable. Since the time allotted to the study of algebra was equal for the four Grade X classes and the

TABLE CIX

COMPARISON RECORD FOR GRADE X CLASSES CONCERNING VIEWPOINT IN ELEMENTARY ALGEBRA

Description of Test	Page	Subject Matter Examined in Test	Score	Mean Score			
				XB	XC	XD	XF
Breslich Algebra Survey Test (1), IV	143	Deriving Equation Verbal Statement	24	12.53 $\pm$.53	11.07 $\pm$.59	10.75 $\pm$.52	9.53 $\pm$.37
Breslich Algebra Survey Test (1), V B	148	Verbal Statement Formula	5	3.32 $\pm$.10	2.38 $\pm$.16	3.05 $\pm$.13	3.09 $\pm$.13
Breslich Algebra Survey Test (1), V C	150	Tabular and Graphical Correspondence	4	2.41 $\pm$.13	1.14 $\pm$.14	1.72 $\pm$.13	1.47 $\pm$.12
Breslich Algebra Survey Test (2), IV	173	Deriving Equation Verbal Statement	10	5.22 $\pm$.23	3.70 $\pm$.19	5.68 $\pm$.17	5.14 $\pm$.14
Breslich Algebra Survey Test (2), VI A	180	Evaluating Algebraic Expressions	10	8.50 $\pm$.14	8.09 $\pm$.24	8.55 $\pm$.16	7.24 $\pm$.25
Breslich Algebra Survey Test (2), VI C	184	Tabular and Graphical Correspondence	7	6.40 $\pm$.11	5.00 $\pm$.22	5.80 $\pm$.13	6.21 $\pm$.11
Breslich Functional Thinking Test, II	202	Unfamiliar Formulas	3	2.94 $\pm$.03	2.82 $\pm$.04	2.69 $\pm$.05	2.65 $\pm$.07
Breslich Functional Thinking Test, IV A	217	Tabular and Graphical Correspondence	5	4.97 $\pm$.02	4.78 $\pm$.06	4.97 $\pm$.02	4.92 $\pm$.04
Breslich Functional Thinking Test, IV B	220	Tabular and Graphical Correspondence	7	4.86 $\pm$.06	4.42 $\pm$.14	3.26 $\pm$.29	1.81 $\pm$.38
Breslich Functional Thinking Test, VI A	236	Tabular and Graphical Correspondence	10	9.72 $\pm$.18	9.78 $\pm$.08	7.60 $\pm$.43	4.54 $\pm$.51

authorized topics for study were the same, the extra time for emphasis upon studies with the variable was secured by reducing the time usually taken for manipulative exercises. It is possible that insufficient time was given to mechanical work during the early part of this presentation; certainly the time was much less than that allowed to the control classes for this phase of their algebra, but there is strong evidence that the pupils of XB did not require this extra training with the symbols in the early part of the course as they continue to improve in their responses to mechanical exercises as they advance with their algebra. This is a valuable item of information revealed by this study. The list of tests and scores (Table CX) shows clearly the comparatively low standing for XB in the early set of tests and the improvement in the second set. As stated, the improvement is mainly due to increased speed with responses and not to a decrease in the number of mechanical errors.

It is evident from the records shown in Table XC that the class XB has not always the lowest score but, in general, the record for XB is lower than that for the control classes. The record for XB in the last four tests shows an improvement over the earlier tests.

Appreciation of Change

The pupils of the experimental class have a nice appreciation of change in varying magnitudes as revealed by their answers to tests relating to change. Throughout this presentation of elementary algebra, continued emphasis has been directed toward change, changing quantities, the measure of changing quantities, instruments for measuring change, related changing quantities, and the record of related change by verbal statement, tabular correspondence, graphical representation, formulas, and by laws of variation. This emphasis has secured an improved response to the tests examining for an appreciation of change. It is evident from the objective evidence submitted in Table CXI that all pupils find the formula method of recording change the most difficult of all methods, yet this is the usual method emphasized in the secondary schools, frequently without reference to other standard methods. One of the most valuable items of information revealed by this study is that concerning the natural delight experienced by the pupils of XB in their studies with changing quantities. It is regrettable that such expression of delight cannot be included with their scores. The list of results (shown in Table CXI) from tests concerning appreciation of change is submitted as objective evidence justifying the foregoing conclusions concerning the difference in pupils' success in interpreting and expressing change.

TABLE CXI

COMPARISON RECORD FOR GRADE X CLASSES CONCERNING APPRECIATION
OF CHANGE AND RECORDS OF RELATED CHANGES

Description of Test	Page	Score	Mean Score			
			XB	XC	XD	XF
Breslich Algebra Survey Test (2), VI B .	182	6	4.25 +	3.52 +	3.57 +	3.35 +
Breslich Algebra Survey Test (2), VI C .	184	7	6.40 +	5.00 +	5.80 +	6.21 +
Breslich Functional Thinking Test, III A .	204	7	6.17 +	5.85 +	5.98 +	6.16 +
Breslich Functional Thinking Test, III B .	206	5	4.86 +	4.70 +	4.52 +	4.54 +
Breslich Functional Thinking Test, III C .	208	7	6.17 +	5.67 +	5.95 +	5.40 +
Breslich Functional Thinking Test, III D .	210	4	2.53 +	2.06 +	2.36 +	2.22 +
Breslich Functional Thinking Test, III E .	211	2	0.86 +	0.27 +	0.93 +	0.60 +
Breslich Functional Thinking Test, III F .	213	2	1.75 +	1.27 +	1.57 +	1.78 +
Breslich Functional Thinking Test, III G .	215	6	5.47 +	4.68 +	4.91 +	4.95 +
Breslich Functional Thinking Test, IV A .	217	5	4.97 +	4.78 +	4.97 +	4.92 +
Breslich Functional Thinking Test, IV B .	220	7	4.86 +	4.42 +	3.26 +	1.81 +
Breslich Functional Thinking Test, IV C .	222	4	3.70 +	2.94 +	2.36 +	1.60 +
Breslich Functional Thinking Test, IV D .	224	8	6.36 +	4.51 +	1.55 +	0.95 +
Breslich Functional Thinking Test, VI B .	238	6	4.61 +	3.67 +	1.49 +	1.19 +
			.16	.19	.18	.11
			.11	.22	.13	.11
			.08	.09	.05	.09
			.09	.14	.14	.15
			.11	.12	.10	.11
			.10	.14	.13	.12
			.11	.07	.10	.09
			.07	.10	.08	.06
			.10	.18	.14	.12
			.02	.06	.02	.04
			.06	.14	.29	.38
			.08	.21	.18	.18
			.18	.26	.21	.19
			.13	.12	.18	.21

There are two features of this report to be considered: the splendid record of an appreciation of change achieved by XC, XD, and XF without any direct teaching concerning this fundamental concept in mathematics and science; and the general improvement in the score for the pupils of the experimental class XB resulting from the attempt to utilize this experience with change when presenting the truths of elementary algebra.

Recognition of Dependence

The pupils of the experimental class exhibit, by their responses to the many tests concerning it, a sound appreciation of dependence of one varying quantity upon another, of related varying magnitudes, of the measures of varying quantities, and of elementary functional relations between related quantities or between their measures. They prove themselves equally efficient in their analysis of such numerical relations when expressed by verbal statement, tabular correspondence, graphical record, and formula-equation. Their record on the Breslich Functional Thinking Test is usually superior to the record from the other Grade X classes and it is frequently comparable to that from the Grade XII class, which has had two years' additional experience in mathematics and science. This topic of dependence is present in many exercises of the standard tests used in this study, but the lists included in Table CXII as objective evidence in support of the foregoing conclusions are those which call directly for knowledge concerning dependence. It is obvious that these tests deal with subjects other than elementary algebra and this implies that lessons concerning dependence, presented in elementary algebra, will assist with the development of a general appreciation of dependence which may prove of value in other studies.

A brief glance at the record in Table CXII will show that XB has not always a high score but generally their score is superior to that for the control classes. There are two distinct thoughts to be gathered from this summary: that the classes XC, XD, and XF have shown a nice appreciation of dependence without explicit instruction concerning this concept; and that the pupils of the experimental class XB exhibit, in general, improvement in score which is due to the continued emphasis upon dependence during the presentation of their algebra.

The Concept of Functional Correspondences

Throughout this presentation of elementary algebra, the pupils of XB were taught to regard the functional correspondence as the fundamental instrument and concept for this subject. They were guided to study, measure, and compare changing quantities, to

TABLE CXII

COMPARISON RECORD FOR GRADE X CLASSES CONCERNING RECOGNITION
OF DEPENDENCE OF ONE QUANTITY UPON ANOTHER

Description of Test	Page	Score	Mean Score					
			XB	XC	XD	XF		
Breslich Algebra Survey Test (1), V A . .	145	18	10.71	.36	13.10	.36	11.94	.40
Breslich Algebra Survey Test (1), V C . .	150	4	2.41	.13	1.14	.14	1.47	.25
Breslich Algebra Survey Test (2), VI A . .	180	10	8.50	.14	8.09	.24	7.24	.12
Breslich Algebra Survey Test (2), VI C . .	184	7	6.40	.11	5.00	.22	6.21	.11
Breslich Algebra Survey Test (2), VI D . .	187	8	4.75	.23	5.15	.24	5.46	.17
Breslich Functional Thinking Test, I A . .	196	4	4.00	.00	4.00	.00	4.00	.00
Breslich Functional Thinking Test, I B . .	198	6	5.47	.08	5.18	.09	5.57	.06
Breslich Functional Thinking Test, I C . .	200	7	6.64	.06	6.27	.12	6.41	.09
Breslich Functional Thinking Test, II A . .	202	3	2.94	.03	2.82	.04	2.69	.07
Breslich Functional Thinking Test, IV A . .	217	5	4.97	.02	4.78	.06	4.97	.04
Breslich Functional Thinking Test, IV B . .	220	7	4.86	.06	4.42	.14	3.26	.37

think about classes of objects, to work with sets of numbers, to form correspondences with related sets of numbers, to study numerical laws relating particular pairs of numbers in correspondence, and to describe such number relations by standard methods. The description of such numerical laws led naturally to the study of variables, formulas, algebraic expressions, and geometric symbols. The functional correspondence gave the pupils their early ideas concerning the formula, conditional equation, and algebraic expression. The results (Table CXIII) from tests which examine the pupils in their appreciation of functional correspondences are submitted as objective evidence that the pupils of the experimental class have gained by this additional training. It is worthy of note that, while the experimental class XB has a superior score with tabular and graphical correspondences, they do not always excel in tests dealing with correspondences described by formula. This result is due to the fact that the control classes received additional training with the formula at the expense of instruction with the more general treatment of functional correspondences.

Just here, it seems appropriate to outline the great number of important mathematical concepts which have their origin in a functional concept. It consists of two (or more) related ranges of numbers. It is the home of the variable and the function. The function is a dependent variable. It involves a numerical relation between the variables which may be expressed by various methods: by word, table, graph, formula, and so on. When there is a formula and known particular values for the independent variable, the numerical relation inherent to the formula is used to find the corresponding values for the dependent variable or the function. Should special values of the function be known, a conditioned equation appears and this is solved to find the corresponding values for the independent variable. Truly the functional correspondence has a great future in Unified Mathematics.

The scores in Table CXIII indicate quite clearly that all Grade X classes have learned something of functional correspondences as they studied mathematics and science. The experimental class exhibits a superior score in all of the tests in Table CXIII except some dealing with formulas. This is a direct result of the modified presentation of their elementary algebra. It should be mentioned here that the control classes would not know of the general term, functional correspondence, which is common to the subject matter tested in the list of exercises in the table.

TABLE CXIII

COMPARISON RECORD FOR GRADE X CLASSES CONCERNING APPRECIATION OF FUNCTIONAL CORRESPONDENCE

Description of Test	Page	Score	Mean Score			
			XB	XC	XD	XF
Breslich Algebra Survey Test (1), V A . .	145	18	10.71 + .36	13.10 + .36	12.88 + .36	11.94 + .40
Breslich Algebra Survey Test (1), V C . .	150	4	2.41 + .13	1.14 + .14	1.72 + .13	1.47 + .12
Breslich Algebra Survey Test (2), VI A . .	180	10	8.50 + .14	8.09 + .24	8.55 + .16	7.24 + .25
Breslich Algebra Survey Test (2), VI C . .	184	7	6.40 + .11	5.00 + .22	5.80 + .13	6.21 + .11
Breslich Algebra Survey Test (2), VI D . .	187	8	4.75 + .23	5.15 + .24	4.88 + .23	5.46 + .17
Breslich Functional Thinking Test, III B	206	5	4.86 + .09	4.70 + .14	4.52 + .14	4.54 + .15
Breslich Functional Thinking Test, III C	208	7	6.17 + .11	5.67 + .12	5.95 + .10	5.40 + .11
Breslich Functional Thinking Test, III D	210	4	2.53 + .10	2.06 + .14	2.36 + .13	2.22 + .12
Breslich Functional Thinking Test, III E	211	2	0.86 + .11	0.27 + .07	0.93 + .10	0.60 + .10
Breslich Functional Thinking Test, III F	213	2	1.75 + .07	1.27 + .10	1.57 + .08	1.78 + .06
Breslich Functional Thinking Test, III G	215	6	5.47 + .10	4.68 + .18	4.91 + .14	4.95 + .12
Breslich Functional Thinking Test, IV A . .	217	5	4.97 + .02	4.78 + .06	4.97 + .02	4.92 + .04
Breslich Functional Thinking Test, IV B . .	220	7	4.86 + .06	4.42 + .14	3.26 + .29	1.81 + .38
Breslich Functional Thinking Test, IV C . .	222	4	3.70 + .08	2.94 + .21	2.36 + .18	1.60 + .18
Breslich Functional Thinking Test, IV D . .	224	8	6.36 + .18	4.51 + .26	1.55 + .21	0.95 + .19
Breslich Functional Thinking Test, V E . .	233	13	9.14 + .25	9.33 + .26	7.90 + .24	9.76 + .17

Improved Rate of Learning as the Course in Algebra Develops

There is evidence in the responses to the tests that the pupils of XB continue to gain in algebraic generalizations and abilities throughout the two years of experimentation. As stated on page 128 of this report, they were promoted to Grade XI in June, 1933, and in June, 1934, they wrote the Provincial Examination in Elementary Algebra for Grade XI. Their record compares favorably with the record for the control classes. The inference is that the modified presentation of elementary algebra prepared them for their more advanced studies in algebra as well as the usual preparatory course in algebra. Their scores on the Breslich Algebra Survey Test, Second Semester, are, in general, higher than those for the corresponding tests for the First Semester, given six months earlier. This is not equally true for the control classes. For example, XF scored well in Algebraic Processes on the Breslich Algebra Survey Test, First Semester, but secured a much lower score in the more difficult corresponding test for the Second Semester. As stated, the low score for XB on those tests of the Breslich Algebra Survey Test, First Semester, dealing with manipulative work may be partly explained by the delay in emphasis upon this phase of elementary algebra; the improved score in the second test originates in improved understandings and abilities which develop as the pupils advance in the subject. The inference is that pupils do such manipulative work with the symbols better if they understand the principles upon which such exercises rest. The results (Table CXIV) from the testing program are submitted as evidence of the foregoing statement concerning the progress of the experimental class. The fact that the two Survey Tests were administered with a six months' interval permits this estimate of progress as well as one of achievement.

Again it is obvious that XB does not always show a superior score for the second test over the score for the first test, but in general there is a steady gain in the corresponding scores for the second test. This rate of learning is discussed from a slightly different viewpoint in Chapter VII, page 264, and the comparison record favors the experimental class.

Functional Thinking

As stated in Chapter III, page 34, of this report, the mental activity known as Functional Thinking may be analyzed into three divisions, which are comparable to those three principles of learning enunciated by Spearman.

1. "Any lived experience tends to evoke immediately a

TABLE CXIV

COMPARISON RECORD FOR GRADE X CLASSES CONCERNING PROGRESS IN ALGEBRAIC THOUGHT AND ABILITIES

Description of Test	Score	Mean Score			
		XB	XC	XD	XF
Algebraic Concepts					
Breslich Algebra Survey Test (1)	20.00	11.41 $\pm$.41	9.45 $\pm$.58	9.84 $\pm$.41	7.76 $\pm$.41
Breslich Algebra Survey Test (2)	16.50	12.67 $\pm$.21	10.43 $\pm$.24	9.33 $\pm$.12	9.14 $\pm$.14
Algebraic Processes					
Breslich Algebra Survey Test (1)	19.00	13.53 $\pm$.21	14.95 $\pm$.23	15.22 $\pm$.13	16.16 $\pm$.14
Breslich Algebra Survey Test (2)	13.00	9.11 $\pm$.24	9.18 $\pm$.22	8.81 $\pm$.18	7.50 $\pm$.17
Solving Equations					
Breslich Algebra Survey Test (1)	18.00	12.60 $\pm$.29	14.00 $\pm$.41	14.00 $\pm$.23	14.36 $\pm$.31
Breslich Algebra Survey Test (2)	34.00	18.94 $\pm$.34	21.76 $\pm$.41	19.79 $\pm$.27	17.02 $\pm$.24
Deriving Equations					
Breslich Algebra Survey Test (1)	24.00	12.53 $\pm$.53	11.07 $\pm$.58	10.75 $\pm$.52	9.53 $\pm$.37
Breslich Algebra Survey Test (2)	10.00	5.22 $\pm$.23	3.70 $\pm$.19	5.68 $\pm$.17	5.14 $\pm$.14
Evaluating Formulas					
Breslich Algebra Survey Test (1)	18.00	10.71 $\pm$.36	13.10 $\pm$.36	12.88 $\pm$.36	11.94 $\pm$.40
Breslich Algebra Survey Test (2)	10.00	8.50 $\pm$.14	8.09 $\pm$.24	8.55 $\pm$.16	7.24 $\pm$.25
Interpreting Graphs					
Breslich Algebra Survey Test (1)	4.00	2.41 $\pm$.13	1.14 $\pm$.14	1.72 $\pm$.13	1.47 $\pm$.12
Breslich Algebra Survey Test (2)	7.00	6.40 $\pm$.11	5.00 $\pm$.22	5.80 $\pm$.13	6.21 $\pm$.11
Fractions					
Breslich Algebra Survey Test (1)	8.00	1.24 $\pm$.16	2.14 $\pm$.23	0.46 $\pm$.10	0.30 $\pm$.11
Breslich Algebra Survey Test (2)	27.00	15.50 $\pm$.63	17.82 $\pm$.51	15.43 $\pm$.47	14.27 $\pm$.51

knowing of its character and its experience."¹

2. "The mentally presenting of any two (or more) characters tends to evoke immediately a knowing of relation between them."²

3. "The presenting of any character together with any relation tends to evoke immediately a knowing of the correlative character."³

To engage in functional thinking, the pupil must apprehend his experience with two (or more) related varying magnitudes, as in the Breslich Functional Thinking Test, I A; or with the measures of two related varying magnitudes, Breslich Functional Thinking Test, III B; or with two given sets of numbers arranged in a functional correspondence, Breslich Functional Thinking Test, V E, exercises 10-13. With such a functional correspondence under active attention, the pupil enters upon a second stage of functional thinking, which is closely related to either one of the mental activities described by Spearman in his Second and Third Principles of Cognition. This relation is pointed out in Chapter III, page 34. The pupils of XB have been encouraged to engage in functional thinking, that is, they have been guided to consider changing quantities, the measures of changing quantities, related sets of numbers arranged in functional correspondences, variables, functions, the representation of functional correspondences, variation, and so on. The pupils of the Grade X control classes have learned incidentally of many of these principles as they worked with their mathematics, science, history, and such other studies presenting discussion concerning related changing quantities. As a result, the score for XB on the Breslich Functional Thinking Test is superior to that from the other Grade X classes. It should be pointed out here that all the exercises in the Breslich Algebra Survey Test and the majority of those exercises employed in the Breslich Functional Thinking Test are selected from the subject matter of Grades IX and X mathematics. Hence it is assumed throughout this study that the difference in responses to these exercises from the pupils in the several Grade X classes is due to the different method employed when presenting the material tested by the exercises. The scores on material not presented to all Grade X classes equally is not included in the Comparison Record given in Table CXV. For example, in the topic "Variation," only the score for Part VI A is included.

Two truths are immediately apparent from the summary in

¹C. A. Spearman, The Nature of Intelligence and The Principles of Cognition, p. 48. London: Macmillan Co., 1923.

²Ibid., p. 63.

³Ibid., p. 91.

the table: one is the splendid record achieved by the Grade X control classes with little, if any, direct teaching in this phase of mathematical learning; and the other is the uniformly superior score for the experimental class, due to additional teaching and exercises in functional thinking.

Another comparison of responses may be found if one reviews the scores for two exercises from the Breslich Functional Thinking Test, viz., Part II, exercise 3, and Part III A, exercise 6. Here all Grade X classes meet an entirely new exercise examining the pupils in their appreciation of a functional correspondence and requiring them to engage in a phase of functional thinking. The scores for these two exercises have been analyzed in detail and presented in Chapter VII, page 278. These scores reveal a decided gain for the experimental class.

Fundamental Concepts

The pupils of the experimental class XB submitted better responses to those parts of the Breslich Algebra Survey Tests which examined them in their appreciation of a carefully selected sample of the fundamental concepts of elementary algebra. This is in accord with expectation when this presentation of elementary algebra was designed. Throughout the teaching program, much more time than is usual in an introductory course was given to illustrations and discussions concerning the fundamental concepts in functional thinking, such as, the appreciation of change, dependence, variable, function, functional correspondence, formula, equation, and so on. The results have been estimated and presented elsewhere in this chapter.

Table CXVI presents two thoughts: (1) The scores for all classes are lower than those in other divisions of these tests as, for example, in the tests on Algebraic Processes. The inference is that pupils learn to do the mechanical work more readily than to master the fundamental principles in elementary algebra. (2) All classes improved their scores for the second test; the inference is that an appreciation of the underlying theory develops as the pupils advance in the subject. XB's score is much better for both tests.

Analysis and Generalization

As a result of this survey of answers to exercises which constitute the several tests in elementary algebra and functional thinking, one is persuaded that pupils of Grade X analyze better than they generalize. This statement applies to the pupils of XB as well as to the pupils of the control units. Nunn writes of analysis and generalization,

TABLE CXV
COMPARISON RECORD FOR GRADE X CLASSES FOR THE BRESLICH FUNCTIONAL THINKING TEST

Description of Test	Score	Mean Score			
		XB	XC	XD	XF
Recognition of Relationships	20	19.00 ± .11	18.27 ± .17	18.64 ± .15	18.59 ± .15
Related Changes	33	27.53 ± .35	24.61 ± .44	25.40 ± .42	26.65 ± .32
Interpreting Graphs	24	21.90 ± .32	18.66 ± .45	14.19 ± .58	11.24 ± .57
Expressing Relationships	40	34.10 ± .38	32.40 ± .57	32.64 ± .39	33.65 ± .31
Variation, Part VI A (only)	10	9.72 ± .18	9.78 ± .18	7.60 ± .43	4.54 ± .51

TABLE CXVI
COMPARISON RECORD FOR GRADE X CLASSES FOR FUNDAMENTAL CONCEPTS IN ELEMENTARY ALGEBRA

Description of Test	Score	Mean Score			
		XB	XC	XD	XF
Breslich Algebra Survey Test (1)	20.00	11.41 ± .41	9.45 ± .58	9.84 ± .41	7.76 ± .41
Breslich Algebra Survey Test (2)	16.50	12.67 ± .	12.67 ± .	10.43 ± .	9.14 ± .

"Strictly speaking the terms analysis and generalization refer to two distinct mental movements. In the former I bring to light the essential process concealed in a particular or accidental numerical garb. In the latter I recognize that this process may be followed identically in solving all problems of the same type. The distinction is not a trivial one but demands the teacher's serious attention."¹

When the pupils are given a set of particular numerical relations between corresponding numbers in a functional correspondence and requested to examine these for evidence of a general law, they, quite frequently, succeed with the exercise in analysis and describe the numerical relation by their own verbal statements in such a manner as to leave little doubt concerning their successful discovery of the numerical relation. They have analyzed for common elements. They have perceived the underlying relating principles. They take much pleasure in such exercises, whether presented in algebraic problems or in functional correspondences expressed in verbal statement, table, or graph. Synthesis consequent to their analysis appears to be more difficult. The pupils find meditation upon the results from analysis less attractive and consequently more difficult. It demands attention to their apprehended experiences with numerical relations. This appears to be less objective in form than the study of the original set of particular arithmetical exercises presented for analysis from a functional correspondence. Consequently, careful attention to the findings from analysis and the consequent synthesis of their results into a generalization does not interest all pupils equally well. The inference is that the curriculum must guide and assist the pupils at this critical stage of development in mathematical thought. It seems to demand a more advanced phase of functional thinking. The Comparison Record in Table CXVII is offered as objective evidence that Grade X pupils succeed with analysis. No objective evidence is offered in support of statements concerning generalization, as the tests do not examine directly for this.

The report in Table CXVII shows that XB has a higher score in these tests requiring the exercise of analysis than the other Grade X classes. The lower score for Part IV, C and D, tends to confirm the statements about generalization; the representation of functional relations by graph is the same for Part IV, A, B, C, and D, but the questions concerning "change" give difficulty because the pupils have not fully grasped the idea of graphical representation of related corresponding numbers.

¹T. P. Nunn, The Teaching of Algebra, p. 2. London: Longmans, Green & Co., 1914.

TABLE CXVII
COMPARISON RECORD FOR GRADE X CLASSES DEALING WITH
ANALYSIS OF PROBLEMS FOR FORMULA-EQUATIONS

Description of Test	Score	Mean Score			
		XB	XC	XD	XF
Breslich Algebra Survey Test (1), IV	24	12.53	11.07	10.75	9.53
Breslich Algebra Survey Test (1), V B	5	3.32	2.38	3.05	3.07
Breslich Algebra Survey Test (2), IV A	10	5.22	3.70	5.68	5.14
Breslich Functional Thinking Test, II	3	2.94	2.82	2.69	2.65
Breslich Functional Thinking Test, IV A	5	4.97	4.78	4.97	4.92
Breslich Functional Thinking Test, IV B	7	4.86	4.42	3.26	1.81
Breslich Functional Thinking Test, IV C	4	3.70	2.94	2.37	1.60
Breslich Functional Thinking Test, IV D	8	6.36	4.51	1.55	0.95
Breslich Functional Thinking Test, V E	13	9.14	9.33	7.90	9.76

Elementary Algebra as a Social Institution

When the principles of elementary algebra develop naturally from studies with functional correspondences, the subject promises to become of greater value as a social institution in that it would tend to satisfy an increasingly urgent social need for further information and education concerning change and related changing quantities. The present course in elementary algebra is designed about the formula and the equation, but modern social needs are wider than these; they demand an appreciation of many methods of recording and expressing functional relations other than by the specialized formula and equation. A survey of the references to functional correspondences appearing in the Chicago Daily Tribune for 1929 was completed and the results were presented in this report in Chapter I, p. 5. A brief review will reveal the varied nature of the references; they contain mention of functional correspondence described by verbal statements, by tabular reports, by graphical records, and by a few formulas. This may be regarded as objective evidence that the social need is one of further education concerning functional relations and their varied forms of expression by words, table, graph, plan, film, formula-equation, and other similar standard and approved forms. The specialized studies with the formula and the equation cannot provide the pupils of the secondary schools immediately with this general outlook upon numerical relations between varying magnitudes. This investigation is undertaken to explain and demonstrate how an introductory course in elementary algebra may originate from a study of functional correspondences and thereby improve this subject as a social institution without weakening the present established course as an approach to more advanced studies in natural science and mathematics. C. H. Judd writes of mathematics in social education:

"Mathematics is a body of experience through the use of which society aims to transform the pupil from what he is in his uneducated state to what he should become that he may be a force in the social world.

.....
"If teachers could comprehend the essentially social character of all teaching they would not be satisfied to give their classes the formal instruction which is frequently found in the schools of Today."¹

This general point of view for elementary algebra does not

¹C. H. Judd, Psychology of Secondary Education, pp. 375-76. Boston: Ginn & Co., 1927.

detract from the importance of this subject as an approach to more advanced studies in this field. The pupils of the experimental class XB, at the close of the two years with this modified presentation of algebra for Grades IX and X, were promoted to Grade XI. One year later, in June, 1934, they wrote the Provincial Examination in Grade XI Algebra and their score compares favorably with that secured by the Grade X control classes, who were taught the authorized course (see page 128). In addition to the principles of Elementary Algebra, these pupils have been introduced to many introductory ideas for the calculus. Thus, by leading the pupils through concrete experience with changing quantities to the generalizations underlying elementary algebra, we have secured that which mathematicians and scientists desire, an introduction of the calculus into the secondary schools.

The pupils of the experimental class XB have been encouraged in functional thinking throughout the presentation of their elementary algebra. This mental activity involves some of the most important principles of learning. There is some evidence that such encouragement in one branch of thought stimulates the exercise of similar mental activities in other subjects which require the consideration of change and an appreciation of related changing factors, for example, in geometry, science, history, geography, economics, and so on.

Conclusion

After a survey of the results from this experimental teaching and testing program in the presentation of elementary algebra as described in this report, my conclusion is that the truths of General Introductory Mathematics should grow naturally from considerations of change, varying magnitudes, the variable, and functional correspondences. Such dynamic studies provide the warp of mathematical education. The subjects of arithmetic, algebra, geometry, trigonometry, analytical geometry, calculus, and other branches of elementary mathematics each with its own peculiar material supply the woof which binds, shapes, and colors the pupil's ever widening and ever deepening appreciations of numerical relationship into the various patterns and fabrics of mathematical thought. Klein describes this essential warp as "The Soul of Mathematics."

